SOLUTION OF A LINEARIZED MODEL OF HEISENBERG'S FUNDAMENTAL EQUATION I

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ABSTRACT. Heisenberg's unsolved fundamental equation of the universe [12, 13] has a coupling constant l which has the dimension of length [L]. We consider a linearized version of Heisenberg's fundamental equation which also contains a coupling constant l with the dimension of a length and we solve this equation in the framework of a relativistic quantum field theory with a fundamental length ℓ in the sense of our recently developed theory [2] and show that then one has $\ell = l/(\sqrt{2}\pi)$. This is done in two parts. In this first part we use path integral methods (and nonstandard analysis) to calculate all Schwinger- and all Wightman- functions of this model, as tempered ultrahyperfunctions and verify some of the defining conditions of a relativistic quantum field theory with a fundamental length, FLQFT for short. As an important intermediate step the convergence of the lattice approximations for a free scalar field and for a Dirac field is shown.

The second part completes the verification of the defining conditions of FLQFT and offers an alternative way to calculate all Wightman functions of the theory.

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1. Introduction

1.1. **Heisenberg's fundamental equation.** The basic relativistic equation of quantum mechanics called Dirac equation

$$i\frac{\hbar}{c}\gamma_\mu\frac{\partial}{\partial x_\mu}\psi(x) - m\psi(x) = 0,\ \ x_0 = ct, x_1 = x, x_2 = y, x_3 = z \qquad (1.1)$$

contains a constants c (velocity of light) which is the fundamental constant in relativity theory, and Planck's constant $h = 2\pi\hbar$ which is the fundamental constant in quantum mechanics. The dimension of c is $[\mathrm{LT}^{-1}]$ and that of h is $[\mathrm{ML}^2\mathrm{T}^{-1}]$. W. Heisenberg thought that a fundamental equation of Physics must also contain a constant l with the dimension of length [L]. If such a constant l is introduced, then the dimensions of any other quantity can be expressed in terms of combinations of the basic constants c, h and l, e.g., time $[\mathrm{T}] = [\mathrm{L}]/[\mathrm{LT}^{-1}]$, or mass as $[\mathrm{M}] = [\mathrm{ML}^2\mathrm{T}^{-1}]/([\mathrm{LT}^{-1}][\mathrm{L}])$.

In 1958, Heisenberg and Pauli introduced the equation

$$\frac{\hbar}{c}\gamma_\mu\frac{\partial}{\partial x_\mu}\psi(x) \pm l^2\gamma_\mu\gamma_5\psi(x)\bar{\psi}(x)\gamma^\mu\gamma_5\psi(x) = 0, \qquad (1.2)$$

which was later called the *equation of the universe* and studied in [8, 13]. The constant l has the dimension [L] and is called the fundamental length of the theory.

Unfortunately, nobody has been able to solve this equation. At present, even in the more advanced framework of ultra-hyperfunction quantum field theory, we do not see how this equation could be solved. Accordingly we study a linearized version of this equations which inherits the important property of a fundamental length l and which first has been studied by Okubo [22]. This linearized version is solvable in the sense of classical field theory, i.e., the classical fields $\phi(x)$ and $\psi(x) = \psi'(x)e^{il^2\phi(x)^2}$ solve this system when ϕ is a solution of the Klein-Gordon equation and ψ' a free Dirac field of mass $\tilde{m}$. We write it in the form

$$\begin{cases} \Box\phi(x) + \left(\dfrac{cm}{\hbar}\right)^2\phi(x) = 0 \\ \left(i\dfrac{\hbar}{c}\gamma^\mu\dfrac{\partial}{\partial x^\mu} - \tilde{m}\right)\psi(x) + 2\gamma^\mu l^2\psi(x)\phi(x)\dfrac{\partial\phi(x)}{\partial x^\mu} = 0 \end{cases} \qquad (1.3)$$

and propose to solve the quantized version of these equations in the framework of a relativistic quantum field theory with a fundamental length as proposed recently by the authors [2] by constructing the Schwinger functions of the fields $\phi(x)$ and $\psi(x)$. And we do so by invoking nonstandard analysis and path integral methods. Thus we calculate the Schwinger functions by means of path integrals on the $*$-finite lattice with an infinitesimal spacing. As a result, the Wightman functions (i.e., the Wick rotated Schwinger functions) of the field $\psi(x)$ are not tempered distributions, but an tempered ultra-hyperfunction.

In the following we will work with the natural units $c = \hbar = 1$. Then the system of equations (1.3) reads

$$(\Box + m^2)\phi(x) = 0 \tag{1.4}$$

$$(i\gamma^\mu \partial_\mu - \tilde{m})\psi = 2l^2\gamma^\mu\psi(x)\phi(x)\partial_\mu\phi(x). \tag{1.5}$$

and they are the field equations of the following Lagrangian density:

$$L(x) = L_{Ff}(x) + L_{Fb}(x) + L_I(x), \tag{1.6}$$

$$L_{Ff}(x) = \bar{\psi}(x)(i\gamma_\mu\partial^\mu - \tilde{m})\psi(x), \tag{1.7}$$

$$L_{Fb}(x) = \frac{1}{2}\{(\partial^\mu\phi(x))^2 - m^2\phi(x)^2\}, \tag{1.8}$$

$$L_I(x) = 2l^2(\bar{\psi}(x)\gamma_\mu\psi(x))\phi(x)\partial^\mu\phi(x). \tag{1.9}$$

1.2. **Relativistic quantum field theory with fundamental length (FLQFT).** As indicated above we are going to show that the system (1.4) - (1.5) can be solved in the framework of a relativistic quantum field theory with a fundamental length (FLQFT) as developed in [2]. This theory is essentially a relativistic quantum field theory in the sense of Gårding and Wightman [24] in terms of operator-valued tempered ultra-hyperfunctions instead of operator-valued tempered Schwartz distributions. The localization properties (in co-ordinate space) of tempered ultra-hyperfunctions (for a technical explanation we have to refer to [2]) are very different from those of Fourier hyperfunctions and (tempered) Schwartz distributions. Tempered ultra-hyperfunctions distinguish events in space-time only when their distance from each other is greater than a certain length ℓ (A heuristic explanation of this property is given in [2]). In contrast to this, Fourier hyperfunctions and Schwartz distributions form a sheaf over space-time and thus exhibit essentially classical localization properties. On the other side the Fourier transforms of tempered ultra-hyperfunctions have essentially classical localization properties in energy-momentum space. Accordingly, compared with relativistic quantum field theory in the sense of Gårding and Wightman (abbreviated as QFT), it is the locality condition (condition of local commutativity) which needs a new formulation in FLQFT. Based on the notion of carrier of analytical functionals we proposed and used in [2] the notion of *extended causality* or *extended local commutativity*.

With this notion of extended local commutativity a full set of defining conditions for a relativistic quantum field theory with a fundamental length has been given and such theories have been characterized in terms of a corresponding full set of conditions on their sequences of vacuum expectation values (n-point or Wightman functionals). In addition an explicit model for such a theory is constructed in [2]. This model is the (Wick) exponential of the square of a free massive field ϕ,

i.e., the field

$$\rho(x) =: e^{g\phi(x)^2} := \sum_{n=0}^{\infty} \frac{g^n}{n!} : \phi(x)^{2n} : .$$

The two-point functional of this field is formally

$$(\Omega, \rho(x)\rho(y)\Omega) = [1 - 4g^2 D_m^{(-)}(x-y)^2]^{-1/2}$$

where $D_m^{(-)}(x-y)$ is the two-point functional of the field ϕ, and the fundamental length of this model is

$$\ell = \frac{\sqrt{g}}{\pi\sqrt{2}}.$$

The major achievements of QFT are the proof of the PCT theorem, the relation between spin and statistics and the existence of a scattering matrix. In FLQFT the PCT and the spin-statistics theorems and the existence of a scattering matrix have been proven too.

1.3. **Motivation for FLQFT.** Very briefly we recall our motivation for our version of a relativistic quantum field theory with a fundamental length.

The first question one has to answer is on which level of the theory the fundamental length should be realized.

The established answer to this question is that the fundamental length should be realized on the level of the geometry of the underlying realization of space-time and accordingly the 'standard' approach to a (quantum) field theory with a fundamental length is to invoke non-commutative geometry [4, 26, 5].

We think that it is important to keep as many of the established physical concepts and results based on the traditional realization of space-time as possible and accordingly have proposed in [2] to realize the fundamental length on the level of the primary dynamical quantities of the theory, namely the fields. In this way we can rely directly on the established physical principles (of field theory, relativistic covariance, physical energy-momentum spectrum, quantum physics). As pointed out above then the only change necessary is that of the realization of the locality principle of standard QFT (when the type of generalized functions to be used in this theory is set to be tempered ultra-hyperfunctions). In this way we arrive at a relativistic quantum field theory in which the fundamental length is realized through special localization properties of the fields and in which the major achievements of standard QFT are still valid.

In the second part where we actually prove these localization properties for our solution we give a brief technical explanation of the localization properties of tempered ultra-hyperfunctions (see subsection 1.2).

2. Path integral quantization

As announced we quantize this model by path integral methods. Formally, the time-ordered two point function is calculated as (see [6])

$$\int \bar{\psi}_\alpha(x_1)\psi_\beta(x_2)\exp i\left\{\int_{\mathbb{R}^4} L_I(x)dx\right\} d\mathcal{D}(\psi,\bar{\psi})d\mathcal{G}(\phi)$$
$$\times\left\{\int \exp i\left\{\int_{\mathbb{R}^4} L_I(x)dx\right\} d\mathcal{D}(\psi,\bar{\psi})d\mathcal{G}(\phi)\right\}^{-1},$$
$$d\mathcal{G}(\phi) = \exp i\left\{\int_{\mathbb{R}^4} L_{Fb}(x)dx\right\}\prod_{x\in\mathbb{R}^4} d\phi(x)$$
$$d\mathcal{D}(\psi,\bar{\psi}) = \exp i\left\{\int_{\mathbb{R}^4} L_{Ff}(x)dx\right\}\prod_{x\in\mathbb{R}^4}\prod_{\alpha 1}^{4}\psi_\alpha(x)\bar{\psi}_\alpha(x).$$

All these integrals have a rigorous meaning if the continuum space-time is replaced by a lattice. We will control the transition from the lattice to the continuum limit by methods from non-standard analysis.

For positive integers M, N define $L = MN$ and $\Delta = \sqrt{\pi}/M$. Then the lattice $\Gamma = \Gamma(M,N)$ is

$$\Gamma = \{t = j\Delta; j\in\mathbb{Z}, -L < j \le L\}.$$

The lattice version of the differential operator $-\triangle + m^2$ on $\mathbb{R}^{\Gamma^4} = \mathbb{R}^{4\cdot 2L}$ is the following difference operator on Γ^4:

$$-\triangle + m^2 : \mathbb{R}^{\Gamma^4} \ni \Phi(x) \to$$
$$-\sum_{\mu=0}^{3}\frac{\Phi(x+e_\mu)+\Phi(x-e_\mu)-2\Phi(x)}{\Delta^2} + m^2\Phi(x) \in \mathbb{R}^{\Gamma^4},$$

where e_μ is the vector of length Δ parallel to the μ-th coordinate axis. Let $dG(\Phi)$ be a Gaussian measure on $\mathbb{R}^{4\cdot 2L}$ defined by

$$dG(\Phi) = Ce^{\left\{\frac{1}{2}\sum_{y\in\Gamma^4}\Phi(y)\left[\sum_{\mu=0}^{3}\frac{\Phi(y+e_\mu)+\Phi(y-e_\mu)-2\Phi(y)}{\Delta^2} - m^2\Phi(y)\right]\Delta^4\right\}}\prod_{y\in\Gamma^4} d\Phi(y), \tag{2.1}$$

where C is the normalization constant such that $\int dG(\Phi) = 1$. Note that the exponent of this measure is the (Euclideanized; $x^0 \to -iy^0$, $\boldsymbol{x} \to \boldsymbol{y}$) discretization of the Lagrangian $\int L_{Fb}(x)dx$. For later use we recall the following well-known formulae for Gaussian integrals on $\mathbb{R}^{4\cdot 2L}$ (see [9]).

$$(2\pi)^{-n/2}\sqrt{\det\Lambda}\int e^{i(y,x)}\exp\left[-\frac{1}{2}(x,\Lambda x)\right]dx = \exp\left[-\frac{1}{2}(y,\Lambda^{-1}y)\right] \tag{A}$$

$$(2\pi)^{-n/2}\sqrt{\det\Lambda}\int(x,Ax)\exp\left[-\frac{1}{2}(x,\Lambda x)\right]dx=\mathrm{Tr}(A\Lambda^{-1}) \qquad (B)$$

where $\mathrm{Re}\,\Lambda$ is strictly positive-definite and A is an arbitrary matrix. Note that the path integral on the finite lattice is the usual integral.

Using (B), we calculate the covariance of the measure $dG(\Phi)$. We define a function $\delta(y)$ on Γ^4 by $\delta(y)=\Delta^{-4}$ if $y=0$ otherwise $\delta(y)=0$, i.e., $\delta(y)=\Delta^{-4}\delta_{0,y}$. Then $(-\triangle_x+m^2)\delta(x-y)$ is the kernel function of the operator $-\triangle+m^2$ and $(-\triangle_x+m^2)\delta(x-y)\Delta^4\Delta^4$ corresponds to the matrix Λ of the formulae (A) and (B) since the summation $\sum_{y\in\Gamma^4}$ is always accompanied by Δ^4. The inverse matrix Λ^{-1} corresponds $(-\triangle_x+m^2)^{-1}\delta(x-y)$ (note that there are no additional Δ). In fact, $(-\epsilon\triangle_x+m^2)\delta(x-y)\Delta^4\Delta^4$ for $\epsilon=0$ is $m^2\delta(x-y)\Delta^4\Delta^4$ $=m^2\delta_{0,x-y}\Delta^{-4}\Delta^4\Delta^4$ and its inverse is $m^{-2}\delta_{0,x-y}\Delta^{-4}=m^{-2}\delta(x-y)$. Now we can calculate the covariance of $dG(\Phi)$.

$$\int\Phi(y_1)\Phi(y_2)dG(\Phi)=\Lambda^{-1}_{y_1,y_2}=(-\triangle+m^2)^{-1}(y_1,y_2)=\mathcal{S}_m(y_1-y_2).$$

Using the lattice Fourier transformation, $\mathcal{S}_m(y_1-y_2)$ is representable as follows:

$$\mathcal{S}_m(y_1-y_2)=(2\pi)^{-4}\sum_{p\in\tilde{\Gamma}^4}e^{ip(y_1-y_2)}\left[\sum_{\mu=0}^{3}(2-2\cos p_\mu\Delta)/\Delta^2+m^2\right]^{-1}\eta^4, \qquad (2.2)$$

where the dual lattice $\tilde{\Gamma}$ is given by

$$\tilde{\Gamma}=\{s=j\eta; j\in\mathbb{Z},-L<j\le L\}, \qquad \eta=\sqrt{\pi}/N.$$

It converges, as $M,N\to\infty$, (see the following section) to the two point Schwinger function

$$S_m(y_1-y_2)=(2\pi)^{-4}\int_{\mathbb{R}^4}e^{ip(y_1-y_2)}\left[p^2+m^2\right]^{-1}d^4p. \qquad (2.3)$$

of a neutral scalar field of mass m.

In order to deal with the fermion field Ψ in the system (1.4) - (1.5) we need to do integration over Grassmann algebras, see [1]. Accordingly we define a measure $dD(\Psi^1,\Psi^2)$ on the Grassmann algebra generated by (see [1]) $\{\Psi^1_\alpha(y),\Psi^2_\alpha(y);\alpha=1,\dots,4,\ y\in\Gamma^4\}$:

$$\begin{aligned}dD(\Psi^1,\Psi^2)=&C'e^{-\left\{\sum_{y\in\Gamma^4}\Psi^{2T}(y)\left[\sum_{\mu=0}^{3}\gamma^E_\mu\nabla_\mu+\tilde{m}\right]\Psi^1(y)\Delta^4\right\}}\\&\prod_{y\in\Gamma^4}\prod_{\alpha=1}^{4}d\Psi^1_\alpha(y)d\Psi^2_\alpha(y),\end{aligned} \qquad (2.4)$$

where C' is another normalization constant, and

$$\Psi^1 = (\Psi^1_1, \dots, \Psi^1_4)^T, \; \Psi^2 = (\Psi^2_1, \dots, \Psi^2_4)^T.$$

The matrices γ^E_μ are related to the Pauli matrices σ_j by $(j = 1, 2, 3)$

$$\gamma^E_0 = \gamma_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \; \gamma^E_j = -i\gamma_j = \begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix},$$

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the operators ∇_μ as discrete versions of the corresponding partial derivatives ∂_μ are defined as follows:

$$\nabla_\mu \Psi_k = \begin{cases} \nabla^+_\mu \Psi_k(y) = (\Psi_k(y + e_\mu) - \Psi_k(y))/\Delta & \text{if } \; k = 1, 2, \\ \nabla^-_\mu \Psi_k(y) = (\Psi_k(y) - \Psi_k(y - e_\mu))/\Delta & \text{if } \; k = 3, 4; \end{cases}$$

namely,

$$\nabla_\mu = P_+ \nabla^+_\mu + P_- \nabla^-_\mu, \qquad P_\pm = (1 \pm \gamma^E_0)/2.$$

The idea to replace the partial derivatives in the continuum case by the forward-, respectively backward difference on the lattice as described above, has originally been developed in [21].

Remark 2.1. It is well known that the free fermion theory on the lattice Γ^4 defined by the action

$$\sum_{x \in \Gamma^4} \Psi^2(x) \left(\sum_{\mu=0}^{3} \gamma^E_\mu [\Psi^1(x + e_\mu) - \Psi^1(x - e_\mu)]/2\Delta + m\Psi^1(x) \right) \Delta^4, \tag{2.5}$$

suffers from the doubling problem. Wilson [25] has overcome this problem by adding the term

$$- \sum_{x \in \Gamma^4} \Psi^2(x) \left(\sum_{\mu=0}^{3} [\Psi^1(x + e_\mu) + \Psi^1(x - e_\mu) - 2\Psi^1(x)]/2\Delta \right) \Delta^4$$

to (2.5)

It is also known that the doubling problem is due to the replacement of the partial derivative ∂_μ by the central difference $(\Psi(x + e_\mu) - \Psi(x - e_\mu))/2\Delta$. If we replace ∂_μ by the forward difference ∇^+_μ respectively the backward difference ∇^-_μ as we have suggested above, we have no doubling problems. Concretely, this is implemented in the Fermion Lagangian density (2.5) by choosing the forward difference for the components Ψ_1, Ψ_2 of the Fermi field while the backward difference is used for the remaining components Ψ_3, Ψ_4.

Kogut and Susskind [14] replaced the derivative of the space variable by half of the central difference, i.e., by $(\Psi(x+e_\mu/2)-\Psi(x-e_\mu/2))/\Delta$. Then the doubling problem disappears but we must introduce the even lattice Γ_e and the odd lattice Γ_o and assign the subset $(\Gamma_e \cup \Gamma_o) \times$

$(\Gamma_e \cup \Gamma_o) \times \Gamma_e)$ of $(\Gamma_e \cup \Gamma_o)^3$ to each field component as its domain of definition. For further details about lattice fermion see [15].

In Section 4 we are going to show that the continuum limit of the covariance (two point function of the lattice Dirac field)

$$\mathcal{R}_{\tilde{m};\alpha,\beta}(y_1 - y_2) = \left[\sum_{\mu=0}^{3} \gamma_\mu^E \nabla_\mu + \tilde{m}\right]^{-1}_{\alpha,\beta} (y_1, y_2) = \int \Psi^1_\alpha(y_1)\Psi^2_\beta(y_2) dD(\Psi^1, \Psi^2) \tag{2.6}$$

coincides with the Schwinger function $R_{\tilde{m};\alpha,\beta}$ of the free Dirac field of mass $\tilde{m}$:

$$R_{\tilde{m};\alpha,\beta}(y) = \left\{ -\sum_{\mu=0}^{3} \gamma_\mu^E \left(\frac{\partial}{\partial y_\mu}\right) + \tilde{m} \right\}_{\alpha,\beta} S_{\tilde{m}}(y)$$

where

$$S_{\tilde{m}}(y) = (2\pi)^{-4} \int_{\mathbb{R}^4} e^{ipy}[p^2 + \tilde{m}^2]^{-1} d^4p$$

Remark 2.2. Though nobody seems to doubt the convergence of the lattice approximations $\mathcal{S}_m(y_1 - y_2)$ respectively $\mathcal{R}_{\tilde{m};\alpha,\beta}(y_1 - y_2)$ to their standard continuum forms $S_m(y_1 - y_2)$ respectively $R_{\tilde{m};\alpha,\beta}(y_1 - y_2)$ we could not find a proof. Maybe these convergence proofs are considered to be too tedious, especially in the case of Fermions due to the doubling problem.

In Section 3, we prove this convergence by using nonstandard analysis [23, 7], that is we show that for any infinitely large $M, N \in {}^*\mathbb{N}$, the standard part of $\mathcal{S}_m(y_1 - y_2)$ is $S_m(y_1 - y_2)$.

Remark 2.3. Sometimes, the proof by nonstandard analysis is simpler and clearer than the standard proof. For example, in order to prove $\lim_{n\to\infty} f(n) = \infty$, for a function $f : \mathbb{N} \to \mathbb{N}$, we must show: $\forall M\ \exists N\ \forall n\ (n \geq N \ \Rightarrow f(n) \geq M)$. But in nonstandard analysis, we can use the formula $I(x)$: x is an infinitely large number, and we have only to show $\forall n\ (I(n) \Rightarrow I(f(n)))$. The number of quantifiers is reduced in this nonstandard proof, and thus it is simpler and clearer (in technical terms: the syntactic complexity of the formula is reduced from a Π_3 formula to a Π_1 formula. See [27]).

In Section 4, the continuum limit of $\mathcal{R}_{\tilde{m};\alpha,\beta}(y_1 - y_2)$ is shown to be $R_{\tilde{m};\alpha,\beta}$. As announced our prescription for avoiding the doubling problem works well here.

Note that in sections 3 and 4, convergence of Schwinger functions is meant not in the sense generalized functions but in the sense of functions.

Certainly, readers can skip sections 3 and 4 if they know or accept that these lattice approximations converge to their expected continuum limit.

Our overall strategy is as follows:

(1) Construct the Schwinger functions

$$[1-4l^4\mathcal{S}_m(y_1-y_2)^2]^{-1/2}\mathcal{R}_{\tilde{m}}(y_1-y_2)$$

in the nonstandard universe;

(2) by taking the standard part or continuous limit, we get standard Schwinger functions $[1-4l^4S_m(y_1-y_2)^2]^{-1/2}R_{\tilde{m}}(y_1-y_2)$. Sections 3 resp. 4 treat the continuous limit of $\mathcal{S}_m(y_1-y_2)^2$ resp. $\mathcal{R}_{\tilde{m}}(y_1-y_2)$.

(3) by Wick rotation, we try to define Wightman functions

$$\lim_{\epsilon\to+0}[1-4l^4D_m^{(-)}(x_0-i\epsilon,\boldsymbol{x})^2]^{-1/2}=\lim_{\epsilon\to+0}[1-4l^4S_m(ix_0+\epsilon,\boldsymbol{x})^2]^{-1/2}$$

from Schwinger functions. But unfortunately, for this to give a mathematically well defined generalized functions ϵ cannot be too small, actually ϵ must be greater than $\ell=l/(\sqrt{2}\pi)$;

(4) in this way, the Wightman functions cannot be a tempered distribution but they can be ultra-hyperfunctions which satisfy, as we will prove later, axiom (R0), and ℓ is the fundamental length according to axiom (R3).

Next we describe our strategy of how to deal with the interaction in this model. We define the Euclideanized lattice Lagrangian density $L_I(y)$ which corresponds to the interaction Lagrangian $L_I(x)$ in (1.9) as follows:

$$\begin{aligned}-L_I(y)=\Psi^{2T}(y)e^{il^2\Phi(y)^2}\sum_{\mu=0}^{3}\gamma_\mu^E\\ \times[P_+\Psi^1(y+e_\mu)\{e^{-il^2\Phi(y+e_\mu)^2}-e^{-il^2\Phi(y)^2}\}/\Delta]\\ +P_-\Psi^1(y-e_\mu)\{e^{-il^2\Phi(y)^2}-e^{-il^2\Phi(y-e_\mu)^2}\}/\Delta.\end{aligned}$$

If we replace the differences in this definition by the corresponding partial derivatives (continuous limit) the above Lagrangian density $L_I(y)$ becomes the Euclideanization ($x^0\to -iy^0$, $\boldsymbol{x}\to\boldsymbol{y}$) of $iL_I(x)$ as given in (1.9).

Now we calculate the lattice version of the Schwinger functions of the interacting fields. The two point Schwinger function is

$$\begin{aligned}&\int\Psi_\alpha^1(y_1)\Psi_\beta^2(y_2)\exp\left(\sum_{y\in\Gamma^4}L_I(y)\Delta^4\right)dD(\Psi^1,\Psi^2)dG(\Phi)\\ &\times\left\{\int\exp\left(\sum_{y\in\Gamma^4}L_I(y)\Delta^4\right)dD(\Psi^1,\Psi^2)dG(\Phi)\right\}^{-1}.\end{aligned}\tag{2.7}$$

If we change the variables

$$\Psi^1(y)=e^{il^2\Phi(y)^2}\Psi'^1(y),\ \ \Psi^2(y)=e^{-il^2\Phi(y)^2}\Psi'^2(y),$$

then (2.7) becomes

$$\int e^{il^2\Phi(y_1)^2}\Psi'^1(y_1)e^{-il^2\Phi(y_2)^2}\Psi'^2(y_2)dD(\Psi'^1,\Psi'^2)dG(\Phi)$$

$$= \int \Psi'^1(y_1)\Psi'^2(y_2)dD(\Psi'^1,\Psi'^2)\int e^{il^2\Phi(y_1)^2}e^{-il^2\Phi(y_2)^2}dG(\Phi).$$

As we are going to show the continuum limit of

$$\int \Psi'^1(y_1)\Psi'^2(y_2)dD(\Psi'^1,\Psi'^2)$$

is the two point Schwinger function $R_{\tilde{m};\alpha,\beta}(y_1-y_2)$ of the free Dirac field. Abbreviate $h_\pm = e^{\pm i\pi/4}l$ and observe that the characteristic function of $(h_-\Phi(y_1), h_+\Phi(y_2))$ is

$$\int e^{ith_-\Phi(y_1)}e^{ish_+\Phi(y_2)}dG(\Phi) = \int e^{ith_-\Phi(y_1)+ish_+\Phi(y_2)}dG(\Phi)$$

$$= \exp -\frac{1}{2}\{th_-\mathcal{S}_m(y_1-y_1)th_- + sh_+\mathcal{S}_m(y_2-y_1)th_-$$

$$+th_-\mathcal{S}_m(y_1-y_2)sh_+ + sh_+\mathcal{S}_m(y_2-y_2)sh_+\}$$

$$= \exp -\frac{1}{2}\{2tsl^2\mathcal{S}_m(y_1-y_2) - it^2l^2\mathcal{S}_m(0) + is^2l^2\mathcal{S}_m(0)\}.$$

By using the relation (see formula (A))

$$(2\pi)^{-1/2}\int e^{ith_\pm\Phi(y)}e^{-t^2/2}dt = e^{-h_\pm^2\Phi(y)^2/2} = e^{\mp il^2\Phi(y)^2/2}$$

we find

$$\int e^{il^2\Phi(y_1)^2}e^{-il^2\Phi(y_2)^2}dG(\Phi)$$

$$= (2\pi)^{-1}\int dtdse^{-t^2/2}e^{-s^2/2}\int e^{it\sqrt{2}h_-\Phi(y_1)}e^{i\sqrt{2}sh_+\Phi(y_2)}dG(\Phi)$$

$$= (2\pi)^{-1}\int dtdse^{-t^2/2}e^{-s^2/2}e^{-\left\{2tsl^2\mathcal{S}_m(y_1-y_2)-\left(it^2l^2\mathcal{S}_m(0)-is^2l^2\mathcal{S}_m(0)\right)\right\}}$$

$$= \left[(1-2il^2\mathcal{S}_m(0))(1+2il^2\mathcal{S}_m(0)) - 4l^4\mathcal{S}_m(y_1-y_2)^2\right]^{-1/2}$$

where we used formula (A) of Gaussian integration for $y=0$.

The value of the two-point Schwinger function at the origin in the lattice approximation diverges in the continuum limit, i.e., $\mathcal{S}_m(0) = \mathcal{S}_m(0;N,M) \to \infty$ as $N, M \to \infty$. In fact,

$$\mathcal{S}_m(0) = (2\pi)^{-4}\sum_{p\in\tilde{\Gamma}^4}\left[\sum_{\mu=0}^{3}(2-2\cos p_\mu\Delta)/\Delta^2 + m^2\right]^{-1}(\sqrt{\pi}/N)^4$$

$$\geq (2\pi)^{-4}\sum_{p\in\tilde{\Gamma}^4}\left[\sum_{\mu=0}^{3}4|p|^2/\pi^2 + m^2\right]^{-1}(\sqrt{\pi}/N)^4$$

$$\to (2\pi)^{-4} \int_{|p_\mu| \le \sqrt{\pi} M} \left[\sum_{\mu=0}^{3} 4|p|^2/\pi^2 + m^2 \right]^{-1} d^4 p \ (N \to \infty)$$

$\to \infty$ as $(M \to \infty)$.

As usual we eliminate this divergence by interpreting the above products respectively power series in the sense of Wick products. A compact form to define Wick products is as follows (see [10]):

$$: e^{ith_\pm \Phi(y)} := \sum_{n=0}^{\infty} [: (ith_\pm \Phi(y))^n : /n!] = e^{\mp it^2 l^2 \mathcal{S}_m(0)} e^{ith_\pm \Phi(y)}.$$

Then we have

$$\int : e^{ith_- \Phi(y_1)} :: e^{ish_+ \Phi(y_2)} : dG(\Phi) = \exp - \left\{ 2tsl^2 \mathcal{S}_m(y_1 - y_2) \right\}$$

and

$$\int : e^{il^2 \Phi(y_1)^2} :: e^{-il^2 \Phi(y_2)^2} : dG(\Phi)$$
$$= (2\pi)^{-1} \int dtdse^{-t^2/2} e^{-s^2/2} \int : e^{i\sqrt{2}th_- \Phi(y_1)} :: e^{i\sqrt{2}sh_+ \Phi(y_2)} : dG(\Phi)$$
$$= (2\pi)^{-1} \int dtdse^{-t^2/2} e^{-s^2/2} \exp - \left\{ 2tsl^2 \mathcal{S}_m(y_1 - y_2) \right\}$$
$$= \left[1 - 4l^4 \mathcal{S}_m(y_1 - y_2)^2 \right]^{-1/2}.$$

Thus the two point Schwinger function of the field ψ in lattice approximation is

$$\left[1 - 4l^4 \mathcal{S}_m(y_1 - y_2)^2 \right]^{-1/2} \mathcal{R}_{\tilde{m};\alpha,\beta}(y_1 - y_2),$$

and its continuous limit (Section 4) is

$$\left[1 - 4l^4 S_m(y_1 - y_2)^2 \right]^{-1/2} R_{\tilde{m};\alpha,\beta}(y_1 - y_2).$$

In order to construct the complete theory the system of Schwinger respectively Wightman functions of all orders $n \in \mathbb{N}$ has to be constructed. We show here how all n-point functions of the interacting fields ϕ and ψ can be calculated in the lattice approximation. For the n-point Schwinger functions of the fields $: e^{it_j h_{r_j} \phi(x)} :$ we find

$$\int \prod_{j=1}^{n} : e^{it_j h_{r_j} \Phi(y_j)} : dG(\Phi) = \prod_{j<k} e^{-t_j t_k h_{r_j} h_{r_k} \mathcal{S}_m(y_j - y_k)},$$

where $r_j = +$ or $r_j = -$, and

$$\int \prod_{j=1}^{n} : e^{-(-1)^{r_j} il^2 \Phi(y_j)^2} : dG(\Phi)$$
$$= \int \prod_{j=1}^{n} dt_j \int \prod_{j=1}^{n} : e^{i\sqrt{2} t_j h_{r_j} \Phi(y_j)} : e^{-t_j^2/2} dG(\Phi)$$

$$= \int \prod_{j=1}^{n} dt_j e^{-t_j^2/2} \prod_{j<k} e^{-2\{t_j t_k h_{r_j} h_{r_k} \mathcal{S}_m(y_j - y_k)} = (\det C)^{-1/2},$$

where the matrix $C = (c_{j,k})$ is given by

$$c_{j,j} = 1, \; c_{j,k} = c_{k,j} = 2h_{r_j} h_{r_k} l^2 \mathcal{S}_m(y_j - y_k) \; j < k. \tag{2.8}$$

where again formula (A) has been used.

Similarly, introduce the matrix $A = (a_{j,k})$ by

$$a_{j,j} = 1, \; a_{j,k} = a_{k,j} = 2h_{r_j} h_{r_k} l^2 D_m^{(-)}(x_j - x_k), \; j < k, \tag{2.9}$$

$$D_m^{(-)}(x_0, \boldsymbol{x}) = S_m(ix_0, \boldsymbol{x}). \tag{2.10}$$

Then the n-point Wightman function

$$\langle 0 | \rho^{(1)}(x_1) \cdots \rho^{(n)}(x_n) | 0 \rangle$$

of the field

$$\rho^{(j)}(x_j) =: e^{-(-1)^{r_j} i l^2 \phi(x_j)^2} : \tag{2.11}$$

is the Wick rotation of the Schwinger function (2.8), i.e.,

$$(\det A)^{-1/2}.$$

This field has been studied in some detail in [19], see also [20].

Next, let $\psi_0(x)$ be the free Dirac field of mass $\tilde{m}$ and introduce the field components $\psi^1(x) = \psi_0(x), \; \psi^2(x) = \bar{\psi}_0(x)$. Denote the Wightman function of the free Dirac field $\psi_0(x)$ by

$$\mathcal{W}_{0,\alpha}^r(x_1, \dots, x_n) = (\Omega, \psi_{\alpha_1}^{r_1}(x_1) \cdots \psi_{\alpha_n}^{r_n}(x_n)\Omega)$$

and let $S_{0,\alpha}^r(y_1, \dots, y_n)$ be its Schwinger function, where $r = (r_1, \dots, r_n)$, $\alpha = (\alpha_1, \dots, \alpha_n)$. Then the n-point Schwinger function of $\psi(y)$ is

$$(\det C)^{-1/2} S_{0,\alpha}^r(y_1, \dots, y_n), \tag{2.12}$$

where $\mathcal{S}_m(y_j - y_k)$ of (2.2) is replaced by its continuous limit $S_m(y_j - y_k)$ (2.3) and where the matric C is defined in (2.8).

Similarly, introduce the matrix $A = (a_{j,k})$ as in (2.9). Then the n-point Wightman function $\mathcal{W}_\alpha^r(x_1, \dots, x_n)$ of the field $\psi(x)$ is the Wick rotation of the Schwinger function (2.12), i.e.,

$$\mathcal{W}_\alpha^r(x_1, \dots, x_n) = (\det A)^{-1/2} \mathcal{W}_{0,\alpha}^r(x_1, \dots, x_n). \tag{2.13}$$

In Section 5, we show that the Wightman functions of $\psi(x)$ are not tempered distributions but tempered ultrahyperfunctions which are studied in [11, 18, 2]. The axiom (R0) of reference [2], modified for the case of Dirac fields is verified.

In part II of our investigations of this linearized model of Heisenberg's equation [3], it is shown that the present model satisfies all the axioms of relativistic quantum field theory with a fundamental length.

3. Convergence of the lattice approximation of the two point functions for free scalar fields

For positive integers M, N put $L = MN$ and let Γ be the 1-dimensional lattice

$$\Gamma = \{x = j\Delta; j \in \mathbb{Z}, -L < j \le L\},$$

with spacing $\Delta = \sqrt{\pi}/M$. Its dual lattice

$$\tilde{\Gamma} = \{p = j\eta; j \in \mathbb{Z}, -L < j \le L\}$$

then has the spacing $\eta = \sqrt{\pi}/N$. Let e_μ be the vector parallel to the μ-th coordinate axis with length Δ, and $\nabla_\mu^\pm$ the forward respectively backward difference in direction e_μ defined by

$$\nabla_\mu^+ \Phi(x) = \frac{\Phi(x+e_\mu) - \Phi(x)}{\Delta}, \quad \nabla_\mu^- \Phi(x) = \frac{\Phi(x) - \Phi(x-e_\mu)}{\Delta}.$$

Then we have

$$\nabla_\mu^+ e^{ipx} = \frac{e^{ip_\mu \Delta} - 1}{\Delta} e^{ipx} = i\bar{q}_\mu e^{ipx} \tag{3.1}$$

$$\nabla_\mu^- e^{ipx} = \frac{1 - e^{-ip_\mu \Delta}}{\Delta} e^{ipx} = iq_\mu e^{ipx}, \tag{3.2}$$

where $q_\mu = (1 - e^{-ip_\mu \Delta})/(i\Delta)$. Note that

$$\nabla_\mu^+ \nabla_\mu^- e^{ipx} = -|q_\mu|^2 e^{ipx} = -\frac{2 - 2\cos p_\mu \Delta}{\Delta^2} e^{ipx}.$$

Accordingly we define a linear operator $-\triangle + m^2 = -\sum_{\mu=0}^3 \nabla_\mu^+ \nabla_\mu^- + m^2$ on $\mathbb{R}^{\Gamma^4} = \mathbb{R}^{4\cdot 2L}$ (second order difference operator on the lattice Γ^4) by

$$\begin{aligned} &-\triangle + m^2 : \mathbb{R}^{\Gamma^4} \ni \Phi(x) \to \\ &\qquad -\sum_{\mu=0}^{3} \frac{\Phi(x+e_\mu) + \Phi(x-e_\mu) - 2\Phi(x)}{\Delta^2} + m^2 \Phi(x) \in \mathbb{R}^{\Gamma^4}. \end{aligned}$$

Using lattice Fourier transformation with periodic boundary conditions, i.e.,

$$\tilde{\Phi}(p) = (2\pi)^{-2} \sum_{x \in \Gamma^4} e^{-ipx} \Phi(x) \Delta^4,$$

$$\Phi(x) = (2\pi)^{-2} \sum_{p \in \tilde{\Gamma}^4} e^{ipx} \tilde{\Phi}(p) \eta^4.$$

this operator has the following simple form in terms of the lattice Fourier transform $\tilde{\Phi}(p)$ of $\Phi(x)$:

$$\begin{aligned} \tilde{\Phi}(p) &\to \left(\sum_{\mu=0}^{3} \frac{-e^{ip_\mu \Delta} - e^{-ip_\mu \Delta} + 2}{\Delta^2} + m^2 \right) \tilde{\Phi}(p) \\ &= \left(\sum_{\mu=0}^{3} \frac{2 - 2\cos p_\mu \Delta}{\Delta^2} + m^2 \right) \tilde{\Phi}(p). \end{aligned}$$

Therefore the kernel $K(x,y)$ (the matrix) of the linear operator $-\triangle + m^2$ is:

$$K(x,y) = (2\pi)^{-4} \sum_{p\in\tilde{\Gamma}^4} e^{ip(x-y)} \left(\sum_{\mu=0}^{3} \frac{2-2\cos p_\mu \Delta}{\Delta^2} + m^2 \right) \eta^4.$$

and the kernel of its inverse is accordingly

$$K^{-1}(x,y) = (2\pi)^{-4} \sum_{p\in\tilde{\Gamma}^4} e^{ip(x-y)} \left(\sum_{\mu=0}^{3} \frac{2-2\cos p_\mu \Delta}{\Delta^2} + m^2 \right)^{-1} \eta^4. \tag{3.3}$$

Note that (3.3) can be written as

$$= \frac{1}{(2\pi)^3} \sum_{\boldsymbol{p}\in\tilde{\Gamma}^3} e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \left(\frac{1}{2\pi} \sum_{p_0\in\tilde{\Gamma}} \frac{e^{ip_0(x^0-y^0)}}{\frac{2-\cos p_0\Delta}{\Delta^2} + A(\boldsymbol{p})^2} \eta \right) \eta^3 \tag{3.4}$$

where

$$A(\boldsymbol{p})^2 = m^2 + \sum_{\mu=1}^{3} \frac{2-2\cos p_\mu \Delta}{\Delta^2}$$

Accordingly we calculate and estimate, for $x \in \Gamma$ and some $B \neq 0$ which later will be chosen to equal $A(\boldsymbol{p})$, the one dimensional lattice sum

$$\sum_{p\in\tilde{\Gamma}} \frac{e^{ixp}}{(2-2\cos p\Delta)/\Delta^2 + B^2} \eta. \tag{3.5}$$

The result is:

Proposition 3.1. *Assume $B \neq 0$ and $|\arg B| \leq \pi/4$. Then one has, for all $x \in \Gamma$,*

$$\sum_{p\in\tilde{\Gamma}} \frac{e^{ixp}}{(2-2\cos p\Delta)/\Delta^2 + B^2} \eta = \frac{2\pi\Delta z_+^{-|x|/\Delta}}{z_+ - z_-} \tag{3.6}$$

$$= \frac{2\pi(1+\Delta B[\sqrt{4+\Delta^2 B^2}/2 + \Delta B/2])^{-|x|/\Delta}}{B\sqrt{4+\Delta^2B^2}}, \tag{3.7}$$

with $z_\pm$ given in (3.9).

If $M, N \in {}^*\mathbb{N}$ are infinitely large numbers and, in the case B is infinitely large, $\delta = \Delta B[\sqrt{4+\Delta^2B^2}/2 + \Delta B/2]$ is infinitesimal, then Eq. (3.7) can be continued by

$$= \frac{2\pi e_*^{-B[\sqrt{4+\Delta^2B^2}/2+\Delta B]|x|}}{B\sqrt{4+\Delta^2B^2}} \tag{3.8}$$

for some $e_* \approx e$, which is near $2\pi e^{-B|x|}/2B$.

Proof. In order to evaluate the sum (3.5) we first rewrite it as

$$\sum_{p\in\tilde{\Gamma}} \frac{e^{ixp}}{(2-2\cos p\Delta)/\Delta^2+B^2}\eta = \sum_{p\in\tilde{\Gamma}} \frac{e^{ixp}}{(2-e^{ip\Delta}-e^{-ip\Delta})/\Delta^2+B^2}\eta =$$
$$\sum_{p\in\tilde{\Gamma}} \frac{e^{i(x+\Delta)p}}{(2e^{ip\Delta}-e^{i2p\Delta}-1)/\Delta^2+e^{ip\Delta}B^2}\eta = \sum_{p\in\tilde{\Gamma}} \frac{\Delta^2 e^{ixp} z}{(2z-z^2-1)+\Delta^2 B^2 z}\eta.$$

for $z = e^{ip\Delta}$. For the decomposition into partial fractions we determine the zeros of the denominator, as a function of the complex variable z, i.e., of

$$z^2-(2+\Delta^2B^2)z+1=0.$$

These zeros are

$$z = z_\pm = \frac{2+\Delta^2B^2 \pm \Delta B\sqrt{4+\Delta^2B^2}}{2}; \tag{3.9}$$

and we can write

$$\frac{1}{2z-z^2-1+z\Delta^2B^2} = \frac{1}{z_+-z_-}\left(\frac{1}{z-z_-}-\frac{1}{z-z_+}\right).$$

Under our assumptions for B we know that $\operatorname{Re} z_+ > 1$, $|z_-|<1$ and $z_+\cdot z_- = 1$. This allows us to use a geometric series to evaluate the lattice sum. Under these conditions we get

$$\frac{z}{z-z_-}-\frac{z}{z-z_+} = \frac{1}{1-\frac{z_-}{z}}+\frac{z/z_+}{1-\frac{z}{z_+}} = \sum_{k=0}^{\infty}\left(\frac{z_-}{z}\right)^k + \frac{z}{z_+}\sum_{k=0}^{\infty}\left(\frac{z}{z_+}\right)^k$$

and accordingly the evaluation of the lattice sum is continued by

$$\frac{\Delta^2\eta}{z_+-z_-}\sum_{p\in\tilde{\Gamma}} e^{ixp}\left(\sum_{k=0}^{\infty} z_-^k e^{-ipk\Delta} + \sum_{k=0}^{\infty} z_+^{-k-1} e^{ip(k+1)\Delta}\right) =$$
$$= \frac{\Delta^2\eta}{z_+-z_-}\sum_{k=0}^{\infty}\left(z_-^k\sum_{p\in\tilde{\Gamma}} e^{ixp}e^{-ipk\Delta} + z_+^{-k-1}\sum_{p\in\tilde{\Gamma}} e^{ixp}e^{ip(k+1)\Delta}\right). \tag{3.10}$$

Now observe that lattice points are of the form $x = k_0\Delta$ for some integer k_0, $-L+1 \le k_0 \le L$ while points of the dual lattice have the form $p = j\eta$, $-L+1\le j\le L$. For $m\in\mathbb{Z}$ one has the following cases

$$\sum_{p\in\tilde{\Gamma}} e^{im\Delta p} = \sum_{j=-L+1}^{L} e^{im\Delta j\eta} = \begin{cases} 2L & m=0 \\ e^{im\frac{\pi}{L}(-L+1)}\frac{1-e^{im\frac{\pi}{L}2L}}{1-e^{im\frac{\pi}{L}}} = 0 & m\neq 0. \end{cases}$$

Accordingly (3.10) equals

$$\frac{\Delta^2\eta}{z_+ - z_-}\sum_{k=0}^{\infty}\left(z_-^k 2L\delta_{k_0,k} + z_+^{-k-1}2L\delta_{-k_0,k+1}\right) =$$

$$= \frac{\Delta^2\eta}{z_+ - z_-}2L\left(\theta(x)z_-^{k_0} + z_+^{k_0}\theta(-x)\right) = \frac{\Delta^2\eta}{z_+ - z_-}2Lz_+^{-|x|/\Delta}.$$

By inserting the expression (3.9), the value (3.7) for the one dimensional lattice sum follows.

Now assume that $M, N \in {}^*\mathbb{N}$ are infinitely large numbers. Denote $u = \delta/|\delta|$, where $\delta = \Delta B[\sqrt{4+\Delta^2B^2}/2 + \Delta B/2]$. Then we have

$$\frac{(1+\Delta B[\sqrt{4+\Delta^2B^2}/2+\Delta B/2])^{-|x|/\Delta}}{B\sqrt{4+\Delta^2B^2}} =$$

$$= \frac{[(1+u|\delta|)^{1/|\delta|}]^{-|\delta||x|/\Delta}}{B\sqrt{4+\Delta^2B^2}} = \frac{e_*^{-B[\sqrt{4+\Delta^2B^2}/2+\Delta B/2]|x|}}{B\sqrt{4+\Delta^2B^2}}$$

where we put

$$(1+u|\delta|)^{1/|\delta|} = e_*^u,$$

i.e., $e_* = e^{\frac{1}{u|\delta|}\log(1+u|\delta|)}$ and $\frac{1}{u|\delta|}\log(1+u|\delta|) \approx 1$, if δ is infinitesimally small. □

Remark 3.2. Among other things our calculations for the lattice sum have established that for $x \in \Gamma$,

$$\sum_{p\in\tilde{\Gamma}}\frac{e^{ipx}e^{ip\Delta}}{e^{ip\Delta} - z_-}\Delta\eta = 2L\Delta\eta z_-^{x/\Delta} = 2\pi z_-^{x/\Delta}.$$

The continuum version of this result reads

$$\int_{-\sqrt{\pi}M}^{\sqrt{\pi}M}\frac{\Delta e^{i(x+\Delta)p}}{e^{ip\Delta} - z_-}dp = \int_{|z|=1}\frac{z^{x/\Delta}}{z - z_-}\frac{dz}{i} = 2\pi z_-^{x/\Delta}.$$

It is interesting to note that the summation and the integration give precisely the same value.

Proposition 3.3. *Let $M, N \in {}^*\mathbb{N}$ be infinitely large numbers and $M_0 = \sqrt{M}$. If $|\boldsymbol{p}| \le M_0$, then for $x_0 \in \Gamma$,*

$$(2\pi)^{-4}\sum_{p_0\in\tilde{\Gamma}}e^{ipx}\left(\sum_{\mu=0}^{3}\frac{2-2\cos p_\mu\Delta}{\Delta^2} + m^2\right)^{-1}\eta$$

$$= (2\pi)^{-3}\frac{e^{i\boldsymbol{p}\boldsymbol{x}}e_{**}(\boldsymbol{p})^{-\sqrt{|\boldsymbol{q}|^2+m^2}|x_0|}}{2\sqrt{|\boldsymbol{q}|^2+m^2}}, \tag{3.11}$$

and $e_{**}(\boldsymbol{p}) \approx e$. *If* $|\boldsymbol{p}| \geq M_0$ *then*

$$\left|(2\pi)^{-4} \sum_{p_0 \in \tilde{\Gamma}} e^{ipx} \left(\sum_{\mu=0}^{3} \frac{2 - 2\cos p_\mu \Delta}{\Delta^2} + m^2 \right)^{-1} \eta \right|$$

$$\leq (2\pi)^{-3} 2^{-2M_0|x_0|/\pi} \frac{1}{2^2 M_0/\pi}. \tag{3.12}$$

Proof. Recall the definition of $A(\boldsymbol{p})$ in (3.4). A basic estimate for the cosine, i.e., $\frac{2}{\pi^2}t^2 \leq 1 - \cos t \leq \frac{1}{2}t^2$ for $|t| \leq \pi$, yields

$$m^2 + \frac{4}{\pi^2}\boldsymbol{p}^2 \leq A(\boldsymbol{p})^2 \leq m^2 + \boldsymbol{p}^2,$$

since $|p_\mu \Delta| \leq \pi$ for $p \in \tilde{\Gamma}^4$.

We prepare the application of Proposition 3.1 with $B = A \equiv A(\boldsymbol{p})$ by checking that $\delta = \Delta A[\sqrt{4 + \Delta^2 A^2}/2 + \Delta A/2]$ is infinitesimal. On the basis of the above estimate for $A(\boldsymbol{p})^2$ this is straightforward for $|\boldsymbol{p}| \leq M_0$. Thus (3.11) follows.

Since $A\sqrt{4 + \Delta^2 A^2}$ and $z_+ = (2 + \Delta^2 A^2 + \Delta A\sqrt{4 + \Delta^2 A^2})/2$ are increasing functions of $A \geq 0$, $z_+^{-|x_0|}/A\sqrt{4 + \Delta^2 A^2}$ is a decreasing function of A and thus is estimated from above by its value at the minimum value A_0 for $A(\boldsymbol{p})$ for $|\boldsymbol{p}| > M_0 = \sqrt{M}$. By our estimate for $A(\boldsymbol{p})^2$ it follows $A_0 \leq 2\sqrt{M}$ and again ΔA_0 and $\delta_0 = \Delta A_0[\sqrt{4 + \Delta^2 A_0^2}/2 + \Delta A_0/2]$ are infinitesimal. Hence Proposition 3.1 applies and (3.12) follows from the lower bound $A_0 \geq \frac{2}{\pi} M_0$. $\square$

Next we prepare the evaluation of the 4-dimensional lattice sum by two lemmas.

Lemma 3.4. *Let* $M, N \in {}^*\mathbb{N}$ *be infinitely large numbers. If* $x_0 \in \Gamma$ *is not infinitesimal, then*

$$(2\pi)^{-4} \sum_{p \in \tilde{\Gamma}^4} e^{ipx} \left(\sum_{\mu=0}^{3} \frac{2 - 2\cos p_\mu \Delta}{\Delta^2} + m^2 \right)^{-1} \eta^4$$

$$\approx (2\pi)^{-3} \sum_{\boldsymbol{p} \in \tilde{\Gamma}^3} \frac{e^{i\boldsymbol{px}} e^{-\sqrt{|\boldsymbol{p}|^2 + m^2}|x_0|}}{2\sqrt{|\boldsymbol{p}|^2 + m^2}} \eta^3.$$

Proof. Let $M_0 = \sqrt{M}$ and suppose that $|x_0|$ is not infinitesimal. Then, by Proposition 3.3

$$\left|(2\pi)^{-4} \sum_{\boldsymbol{p} \in \tilde{\Gamma}^3, |\boldsymbol{p}| \geq M_0} \sum_{p_0 \in \tilde{\Gamma}} e^{ipx} \left(\sum_{\mu=0}^{3} \frac{2 - 2\cos p_\mu \Delta}{\Delta^2} + m^2 \right)^{-1} \eta^4 \right|$$

$$\leq (2\pi)^{-3} \sum_{\boldsymbol{p} \in \tilde{\Gamma}^3, |\boldsymbol{p}| \geq M_0} \frac{2^{-\sqrt{2}M_0|x_0|/\pi}}{2\sqrt{2}M_0/\pi} \eta^3 \leq (2\pi)^{-3/2} M^3 \frac{2^{-2\sqrt{M}|x_0|/\pi}}{2^2\sqrt{M}/\pi} \approx 0.$$

Since

$$\left|(2\pi)^{-3}\frac{e_{**}(\boldsymbol{p})^{-\sqrt{|\boldsymbol{q}|^2+m^2}|x_0|}}{2\sqrt{|\boldsymbol{q}|^2+m^2}}\right| \le (2\pi)^{-3}\frac{2^{-2|\boldsymbol{p}||x_0|/\pi}}{4|\boldsymbol{p}|/\pi},$$

for any standard $\epsilon > 0$, there exists a finite $M_1 > 0$ such that

$$\left|(2\pi)^{-3}\sum_{\boldsymbol{p}\in\tilde{\Gamma}^3, M_1\le|\boldsymbol{p}|\le M_0}\frac{e_{**}(\boldsymbol{p})^{-\sqrt{|\boldsymbol{q}|^2+m^2}|x_0|}}{2\sqrt{|\boldsymbol{q}|^2+m^2}}\eta^3\right| < \epsilon$$

if $|x_0|$ is not infinitesimal. This shows that for all $\epsilon > 0$ there exists M_1 such that

$$\left|(2\pi)^{-4}\sum_{\boldsymbol{p}\in\tilde{\Gamma}^3,|\boldsymbol{p}|\ge M_1}\sum_{p_0\in\tilde{\Gamma}} e^{ipx}\left(\sum_{\mu=0}^{3}\frac{2-2\cos p_\mu\Delta}{\Delta^2}+m^2\right)^{-1}\eta^4\right| < \epsilon.$$

We also have

$$\forall\, \epsilon > 0\ \exists\, M_1 \quad \left|\sum_{\boldsymbol{p}\in\tilde{\Gamma}^3,|\boldsymbol{p}|\ge M_1}\frac{e^{-\sqrt{|\boldsymbol{p}|^2+m^2}|x_0|}}{\sqrt{|\boldsymbol{p}|^2+m^2}}\eta^3\right| < \epsilon.$$

Let $M_1 > 0$ be finite. If $|\boldsymbol{p}| \le M_1$, then one has, for some $0 < \theta_\mu = \theta(p_\mu) < 1$,

$$|\boldsymbol{q}|^2 = \sum_{\mu=1}^{3}\frac{2-2\cos p_\mu\Delta}{\Delta^2} = \sum_{\mu=1}^{3}(p_\mu^2 + \frac{\sin\theta_\mu p_\mu\Delta}{3!}p_\mu^3\Delta) \approx \sum_{\mu=1}^{3}p_\mu^2$$

and

$$= \frac{e_{**}(\boldsymbol{p})^{-\sqrt{|\boldsymbol{q}|^2+m^2}|x_0|}}{2\sqrt{|\boldsymbol{q}|^2+m^2}} \approx \frac{e^{-\sqrt{|\boldsymbol{p}|^2+m^2}|x_0|}}{2\sqrt{|\boldsymbol{p}|^2+m^2}}.$$

It follows that

$$\Bigg|(2\pi)^{-3}\sum_{\boldsymbol{p}\in\tilde{\Gamma}^3,|\boldsymbol{p}|\le M_1}\frac{e^{i\boldsymbol{px}}e_{**}(\boldsymbol{p})^{-\sqrt{|\boldsymbol{q}|^2+m^2}|x_0|}}{2\sqrt{|\boldsymbol{q}|^2+m^2}}\eta^3$$
$$-(2\pi)^{-3}\sum_{\boldsymbol{p}\in\tilde{\Gamma}^3,|\boldsymbol{p}|\le M_1}\frac{e^{i\boldsymbol{px}}e^{-\sqrt{|\boldsymbol{p}|^2+m^2}|x_0|}}{2\sqrt{|\boldsymbol{p}|^2+m^2}}\eta^3\Bigg| \approx 0$$

and hence

$$(2\pi)^{-4}\sum_{p\in\tilde{\Gamma}^4}e^{ipx}\left(\sum_{\mu=0}^{3}\frac{2-2\cos p_\mu\Delta}{\Delta^2}+m^2\right)^{-1}\eta^4$$
$$\approx (2\pi)^{-3}\sum_{\boldsymbol{p}\in\tilde{\Gamma}^3}\frac{e^{i\boldsymbol{px}}e^{-\sqrt{|\boldsymbol{p}|^2+m^2}|x_0|}}{2\sqrt{|\boldsymbol{p}|^2+m^2}}\eta^3.$$

$\square$

Lemma 3.5. *Assume that $M, N \in^* \mathbb{N}$ are infinitely large numbers. If $\boldsymbol{x}$ is finite and $|x_0|$ not infinitesimal, then the following lattice sum is infinitesimally close to the expected integral:*

$$(2\pi)^{-3} \sum_{\boldsymbol{p}\in\tilde{\Gamma}^3} \frac{e^{i\boldsymbol{px}} e^{-\sqrt{|\boldsymbol{p}|^2+m^2}|x_0|}}{2\sqrt{|\boldsymbol{p}|^2+m^2}} \eta^3 \approx (2\pi)^{-3} \int_{^*\mathbb{R}^3} \frac{e^{i\boldsymbol{px}} e^{-\sqrt{|\boldsymbol{p}|^2+m^2}|x_0|}}{2\sqrt{|\boldsymbol{p}|^2+m^2}} d\boldsymbol{p}.$$

Proof. For

$$f(x,\boldsymbol{p}) = \frac{e^{i\boldsymbol{px}} e^{-\sqrt{|\boldsymbol{p}|^2+m^2}|x_0|}}{\sqrt{|\boldsymbol{p}|^2+m^2}}$$

calculate

$$\frac{\partial}{\partial p_\mu} f(x,\boldsymbol{p}) =$$

$$-\frac{e^{i\boldsymbol{px}} e^{-\sqrt{|\boldsymbol{p}|^2+m^2}|x_0|} p_\mu}{\sqrt{(|\boldsymbol{p}|^2+m^2)^3}} + \left(i x_\mu - \frac{|x_0| p_\mu}{\sqrt{|\boldsymbol{p}|^2+m^2}} \right) \frac{e^{i\boldsymbol{px}} e^{-\sqrt{|\boldsymbol{p}|^2+m^2}|x_0|}}{\sqrt{|\boldsymbol{p}|^2+m^2}}.$$

and estimate

$$\left| \frac{\partial}{\partial p_\mu} f(x,\boldsymbol{p}) \right| \le e^{-\sqrt{|\boldsymbol{p}|^2+m^2}|x_0|} \left(\frac{|x_0|+|x_\mu|}{\sqrt{|\boldsymbol{p}|^2+m^2}} + \frac{1}{|\boldsymbol{p}|^2+m^2} \right).$$

Therefore the variation of $f(x,\boldsymbol{p})$ on $\prod_{\mu=1}^{3} [p_\mu - \eta/2, p_\mu + \eta/2]$ is smaller than

$$3\frac{\sqrt{\pi}}{N} e^{-\sqrt{|\boldsymbol{p}|^2+m^2}|x_0|} \left(\frac{|x_0|+|x_\mu|}{\sqrt{|\boldsymbol{p}|^2+m^2}} + \frac{1}{|\boldsymbol{p}|^2+m^2} \right).$$

This shows that

$$\left| \sum_{\boldsymbol{p}\in\tilde{\Gamma}^3} f(x,\boldsymbol{p})\eta^3 - \int_{[-\sqrt{\pi}M,\sqrt{\pi}M]^3} f(x,\boldsymbol{p}) d\boldsymbol{p} \right|$$

$$\le 3\frac{\sqrt{\pi}}{N} \int_{[-\sqrt{\pi}M,\sqrt{\pi}M]^3} \frac{(|x_0|+|x_\mu|) e^{-\sqrt{|\boldsymbol{p}|^2+m^2}|x_0|}}{\sqrt{|\boldsymbol{p}|^2+m^2}} d\boldsymbol{p}$$

$$+3\frac{\sqrt{\pi}}{N} \int_{[-\sqrt{\pi}M,\sqrt{\pi}M]^3} \frac{e^{-\sqrt{|\boldsymbol{p}|^2+m^2}|x_0|}}{|\boldsymbol{p}|^2+m^2} d\boldsymbol{p}$$

$$\le 3\frac{\sqrt{\pi}}{N} (|x_0|+|x_\mu|)|e^{-m|x_0|/\sqrt{2}} \int_{[-\sqrt{\pi}M,\sqrt{\pi}M]^3} \frac{e^{-|\boldsymbol{p}||x_0|/\sqrt{2}}}{\sqrt{|\boldsymbol{p}|^2+m^2}} d\boldsymbol{p}$$

$$+3\frac{\sqrt{\pi}}{N} \int_{[-\sqrt{\pi}M,\sqrt{\pi}M]^3} \frac{e^{-\sqrt{|\boldsymbol{p}|^2+m^2}|x_0|}}{|\boldsymbol{p}|^2+m^2} d\boldsymbol{p} \approx 0.$$

Since

$$\int_{|\boldsymbol{p}|\ge\sqrt{\pi}M} f(x,\boldsymbol{p}) d\boldsymbol{p} \approx 0,$$

this lemma is proved. □

By combining Lemmas 3.4 and 3.5 we arrive at the main result of this section.

Theorem 3.6. *Assume that $M, N \in {}^*\mathbb{N}$ are infinitely large numbers. If $x, y \in \Gamma^4$ is finite and $|x_0 - y_0|$ is not infinitesimal, then the lattice sum (3.3) is infinitesimally close to the expected integral*

$$\begin{aligned} K^{-1}(x,y) &= (2\pi)^{-4} \sum_{p \in \tilde{\Gamma}^4} e^{ip(x-y)} \left(\sum_{\mu=0}^{3} \frac{2 - 2\cos p_\mu \Delta}{\Delta^2} + m^2 \right)^{-1} \eta^4 \\ &\approx (2\pi)^{-3} \int_{{}^*\mathbb{R}^3} \frac{e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})} e^{-\sqrt{|\boldsymbol{p}|^2+m^2}|x_0-y_0|}}{2\sqrt{|\boldsymbol{p}|^2+m^2}} d\boldsymbol{p} \end{aligned} \tag{3.13}$$

According to this result, the continuum limit of $K^{-1}(x,y)$ is the two point Schwinger function $S_m(x-y)$ of the free neutral scalar field of mass m. In fact, if x and y are standard real number and $x_0 \neq y_0$ then

$$\begin{aligned} {}^*S_m(x-y) &= \int_{{}^*\mathbb{R}^3} \frac{e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})} e^{-\sqrt{|\boldsymbol{p}|^2+m^2}|x_0-y_0|}}{2\sqrt{|\boldsymbol{p}|^2+m^2}} d\boldsymbol{p} \\ &= \int_{\mathbb{R}^3} \frac{e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})} e^{-\sqrt{|\boldsymbol{p}|^2+m^2}|x_0-y_0|}}{2\sqrt{|\boldsymbol{p}|^2+m^2}} d\boldsymbol{p} = S_m(x-y). \end{aligned}$$

This follows from the transfer principle of nonstandard analysis and means that integrations in both standard universe and nonstandard universe coincide. For finite $x, y \in \Gamma$ such that $x_0 - y_0$ is not infinitesimal, we have

$$\mathcal{S}_m(x-y) = K^{-1}(x,y) \approx {}^*S_m(x-y) \approx {}^*S_m(\operatorname{st} x - \operatorname{st} y) = S_m(\operatorname{st} x - \operatorname{st} y),$$

where $\operatorname{st} x$ is the standard part of x, i.e., the unique standard real number infinitesimally close to x. The two point Wightman function thus is

$$\begin{aligned} &\lim_{\epsilon \to +0} S_m(i(x_0-y_0)+\epsilon, \boldsymbol{x}-\boldsymbol{y}) \\ &= \lim_{\epsilon \to +0} D_m^{(-)}(x_0-y_0-i\epsilon, \boldsymbol{x}-\boldsymbol{y}) = D_m^{(-)}(x_0-y_0, \boldsymbol{x}-\boldsymbol{y}) \\ &= \lim_{\epsilon \to +0} (2\pi)^{-3} \int_{{}^*\mathbb{R}^3} \frac{e^{i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})} e^{-i\sqrt{|\boldsymbol{p}|^2+m^2}(x_0-y_0-i\epsilon)}}{2\sqrt{|\boldsymbol{p}|^2+m^2}} d\boldsymbol{p}. \end{aligned} \tag{3.14}$$

4. Convergence of the lattice approximation of the two point Schwinger function for the free Dirac field

We denote $\Psi(x) = (\Psi_1(x), \ldots, \Psi_4(x))^T$, and recall the notation introduced in Section 2. The discrete version of the Dirac operator then

is

$$\sum_{\mu=0}^{3} \gamma_\mu^E \nabla_\mu + \tilde{m}. \tag{4.1}$$

The kernel of its inverse will be the lattice form of the 2-point Schwinger function for the free Dirac field (compare (2.6)). Naturally, we determine this inverse in analogy to the continuum case and use lattice Fourier transformation instead of the standard Fourier transformation.

The lattice Fourier transformation transforms $\gamma_j^E \nabla_\mu \Psi(x)$ into

$$\begin{pmatrix} 0 & -i\sigma_j \\ i\sigma_j & 0 \end{pmatrix} \begin{pmatrix} (e^{ip_j\Delta}-1)/\Delta & 0 \\ 0 & (1-e^{-ip_j\Delta})/\Delta \end{pmatrix} \tilde{\Psi}(p)$$

$$= \begin{pmatrix} 0 & \sigma_j q_j \\ -\sigma_j \bar{q}_j & 0 \end{pmatrix} \tilde{\Psi}(p)$$

if $j = 1, 2, 3$, with $q_j = -i(1 - e^{-ip_j\Delta})/\Delta$, respectively into

$$\begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} \begin{pmatrix} (e^{ip_0\Delta}-1)/\Delta & 0 \\ 0 & (1-e^{-ip_0\Delta})/\Delta \end{pmatrix} \tilde{\Psi}(p)$$

$$= \begin{pmatrix} -i\sigma_0 q_0 & 0 \\ 0 & i\sigma_0 \bar{q}_0 \end{pmatrix} \tilde{\Psi}(p).$$

if $j = 0$. Thus the Dirac operator

$$\left[\sum_{\mu=0}^{3} \gamma_\mu^E \nabla_\mu + \tilde{m}\right] \Psi(x)$$

is transformed into

$$\begin{pmatrix} i\bar{q}_0 + \tilde{m} & \boldsymbol{\sigma}\cdot\boldsymbol{q} \\ -\boldsymbol{\sigma}\cdot\bar{\boldsymbol{q}} & -iq_0 + \tilde{m} \end{pmatrix} \tilde{\Psi}(p).$$

In order to calculate the inverse of the Dirac operator, in analogy to the continuum case, we calculate first the inverse of the second order differential operator of which the Dirac operator is a factor. Accordingly we determine first this second order operator.

Under lattice Fourier transformation the operator

$$\left[-\gamma_0^E \nabla_0' - \sum_{j=1}^{3} \gamma_j^E \nabla_j + \tilde{m}\right] \Psi(x),$$

where $\nabla_\mu' = P_+ \nabla_\mu^- + P_- \nabla_\mu^+$, is transformed into

$$\begin{pmatrix} -iq_0 + \tilde{m} & -\boldsymbol{\sigma}\cdot\boldsymbol{q} \\ \boldsymbol{\sigma}\cdot\bar{\boldsymbol{q}} & i\bar{q}_0 + \tilde{m} \end{pmatrix} \tilde{\Psi}(p).$$

In order to calculate the composition

$$\begin{pmatrix} i\bar{q}_0 + \tilde{m} & \boldsymbol{\sigma}\cdot\boldsymbol{q} \\ -\boldsymbol{\sigma}\cdot\bar{\boldsymbol{q}} & -iq_0 + \tilde{m} \end{pmatrix} \begin{pmatrix} -iq_0 + \tilde{m} & -\boldsymbol{\sigma}\cdot\boldsymbol{q} \\ \boldsymbol{\sigma}\cdot\bar{\boldsymbol{q}} & i\bar{q}_0 + \tilde{m} \end{pmatrix} \tag{4.2}$$

of the above two operators we introduce the abbreviations $a = i\bar{q}_0 + \tilde{m}$ and $b = \boldsymbol{\sigma} \cdot \boldsymbol{q}$ and find

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} \bar{a} & -b \\ \bar{b} & a \end{pmatrix} = \begin{pmatrix} a\bar{a} + b\bar{b} & 0 \\ 0 & \bar{a}a + \bar{b}b \end{pmatrix} = \begin{pmatrix} K & 0 \\ 0 & K^* \end{pmatrix}$$

with

$$\begin{aligned} K = a\bar{a} + b\bar{b} &= (|q_0|^2 + |\boldsymbol{q}|^2 + \tilde{m}^2 + 2i\tilde{m}(\bar{q}_0 - q_0))\sigma_0 \\ &\quad + (q_1\bar{q}_2 - q_2\bar{q}_1)\sigma_3 + (q_2\bar{q}_3 - q_3\bar{q}_2)\sigma_1 + (q_3\bar{q}_1 - q_1\bar{q}_3)\sigma_2. \end{aligned}$$

We decompose the matrix K into its Hermitian and anti-Hermitian part

$$K = D + 2iE, \quad K^* = D - 2iE$$

where

$$\begin{aligned} D &= (|q_0|^2 + |\boldsymbol{q}|^2 + \tilde{m}^2 + 2i\tilde{m}(\bar{q}_0 - q_0))\sigma_0, \\ E &= (\mathrm{Im}\, q_2\bar{q}_3)\sigma_1 + (\mathrm{Im}\, q_3\bar{q}_1)\sigma_2 + (\mathrm{Im}\, q_1\bar{q}_2)\sigma_3. \end{aligned}$$

Observe that

$$\begin{aligned} i(\bar{q}_0 - q_0) &= 2\mathrm{Im}\, q_0 = 2(1 - \cos p_0\Delta)/\Delta \geq 0, \\ q_j\bar{q}_k - q_k\bar{q}_j &= 2i\mathrm{Im}\,(q_j\bar{q}_k). \end{aligned}$$

Next we calculate the eigenvalues of the Hermitian matrix E from the equation

$$\det(E - \lambda\sigma_0) = \lambda^2 - (\mathrm{Im}\, q_2\bar{q}_3)^2 - (\mathrm{Im}\, q_3\bar{q}_1)^2 - (\mathrm{Im}\, q_1\bar{q}_2)^2 = 0$$

and find

$$\lambda = \pm\sqrt{(\mathrm{Im}\, q_2\bar{q}_3)^2 + (\mathrm{Im}\, q_3\bar{q}_1)^2 + (\mathrm{Im}\, q_1\bar{q}_2)^2} = \pm\rho.$$

There exist orthonormal eigenvectors $x_\pm$ of E, i.e., $Ex_\pm = \pm\rho x_\pm$, which are also eigenvectors of D and thus of K:

$$Dx_\pm = (|q_0|^2 + |\boldsymbol{q}|^2 + \tilde{m}^2 + 2i\tilde{m}(\bar{q}_0 - q_0))x_\pm = \kappa x_\pm$$

and

$$Kx_\pm = (D + 2iE)x_\pm = (\kappa \pm 2i\rho)x_\pm.$$

It follows, for any $\alpha_\pm \in \mathbb{C}$, that $K(\alpha_+ x_+ + \alpha_- x_-) = \alpha_+(\kappa + 2i\rho)x_+ + \alpha_-(\kappa - 2i\rho)x_-$ and therefore $\|Kx\|^2 = (\kappa^2 + 4\rho^2)\|x\|^2$ for all vectors x. Therefore $(\kappa^2 + 4\rho^2)^{-\frac{1}{2}}K$ is a unitary matrix. It also follows that $(\kappa^2 + 4\rho^2)^{-\frac{1}{2}}K^*$ is a unitary matrix too and thus the following relations hold:

$$\begin{aligned} \sqrt{\kappa^2 + 4\rho^2}K^{-1} &= K^*/\sqrt{\kappa^2 + 4\rho^2},\ K^{-1} = K^*/(\kappa^2 + 4\rho^2). \\ \sqrt{\kappa^2 + 4\rho^2}K^{*-1} &= K/\sqrt{\kappa^2 + 4\rho^2},\ K^{*-1} = K/(\kappa^2 + 4\rho^2). \end{aligned}$$

Now it is straightforward to calculate the inverse of the product operator (4.2):

$$\begin{pmatrix} -iq_0 + \tilde{m} & -\boldsymbol{\sigma} \cdot \boldsymbol{q} \\ \boldsymbol{\sigma} \cdot \overline{\boldsymbol{q}} & i\bar{q}_0 + \tilde{m} \end{pmatrix}^{-1} \begin{pmatrix} i\bar{q}_0 + \tilde{m} & \boldsymbol{\sigma} \cdot \boldsymbol{q} \\ -\boldsymbol{\sigma} \cdot \overline{\boldsymbol{q}} & -iq_0 + \tilde{m} \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} K^{-1} & 0 \\ 0 & K^{*-1} \end{pmatrix} = \frac{1}{\kappa^2 + 4\rho^2} \begin{pmatrix} K^* & 0 \\ 0 & K \end{pmatrix},$$

from which the inverse of the Dirac operator in lattice Fourier transformed form is easily calculated as

$$\begin{pmatrix} i\bar{q}_0 + \tilde{m} & \boldsymbol{\sigma} \cdot \boldsymbol{q} \\ -\boldsymbol{\sigma} \cdot \overline{\boldsymbol{q}} & -iq_0 + \tilde{m} \end{pmatrix}^{-1} = \frac{1}{\kappa^2 + 4\rho^2} \begin{pmatrix} -iq_0 + \tilde{m} & -\boldsymbol{\sigma} \cdot \boldsymbol{q} \\ \boldsymbol{\sigma} \cdot \overline{\boldsymbol{q}} & i\bar{q}_0 + \tilde{m} \end{pmatrix} \begin{pmatrix} K^* & 0 \\ 0 & K \end{pmatrix}. \quad (4.3)$$

In the rest of this section we are going to calculate the lattice Fourier transform of this identity and will show that the continuum limit of

$$(2\pi)^{-4} \sum_{p \in \tilde{\Gamma}^4} \frac{e^{ixp}}{\kappa^2 + 4\rho^2} \begin{pmatrix} \tilde{m} & -\boldsymbol{\sigma} \cdot \boldsymbol{q} \\ \boldsymbol{\sigma} \cdot \overline{\boldsymbol{q}} & \tilde{m} \end{pmatrix} \begin{pmatrix} K^* & 0 \\ 0 & K \end{pmatrix} \eta^4 \quad (4.4)$$

equals the well known integral representation of the two-point function. As in the scalar case, the four dimensional lattice sum is evaluated successively, beginning with the sum $\sum_{p_0 \in \tilde{\Gamma}}$. After some preparations, a succession of lemmas will prepare the final result of this section, Theorem 4.11.

In order to calculate the matrix

$$(\kappa^2 + 4\rho^2)^{-1} K = (\kappa^2 + 4\rho^2)^{-1} (\kappa\sigma_0 + 2iE)$$

we determine first the factor

$$\frac{\kappa}{\kappa^2 + 4\rho^2} = \frac{1}{2} \left(\frac{1}{\kappa + 2i\rho} + \frac{1}{\kappa - 2i\rho} \right)$$

and expand $\kappa = |q_0|^2 + |\boldsymbol{q}|^2 + \tilde{m}^2 + 2i\tilde{m}(\bar{q}_0 - q_0) = (1 - 2\tilde{m}\Delta)(2 - 2\cos p_0\Delta)/\Delta^2 + |\boldsymbol{q}|^2 + \tilde{m}^2$.

Next we prepare the evaluation of the sum

$$\frac{1}{1 - 2\tilde{m}\Delta} \sum_{p \in \tilde{\Gamma}} \frac{e^{ixp}}{(2 - 2\cos p\Delta)/\Delta^2 + B_\pm^2} \eta,$$

$$B_\pm^2 = \frac{A^2 \pm i\rho}{1 - 2\tilde{m}\Delta}, \quad A^2 = |\boldsymbol{q}|^2 + \tilde{m}^2$$

with the help of Proposition 3.1 by the following lemma.

Lemma 4.1. *For the quantities introduced above these statements hold:*
a) $0 \le \rho \le |\boldsymbol{q}|^2/\sqrt{2}$, $0 \le \pm \arg B_\pm \le \pi/6$;
b) if $|\boldsymbol{p}| \le M_0 = \sqrt{M}$, *then* $\rho \le \sqrt{\pi}|\boldsymbol{q}|^2/(\sqrt{2}M_0)$ *and* $\arg B_\pm \approx 0$ *if* M *is an infinitely large number.*

Proof. Note that

$$q_k\bar{q}_j\Delta^2 = (e^{ip_j\Delta}-1)(e^{-ip_k\Delta}-1) = e^{i(p_j-p_k)\Delta} - e^{ip_j\Delta} - e^{-ip_k\Delta} + 1,$$

$$\begin{aligned}\operatorname{Im} q_k\bar{q}_j\Delta^2 &= \sin(p_j-p_k)\Delta + \sin p_j\Delta + \sin p_k\Delta\\ &= \sin p_j\Delta\cos p_k\Delta - \cos p_j\Delta\sin p_k\Delta - \sin p_j\Delta + \sin p_k\Delta\\ &= \sin p_k\Delta(1-\cos p_j\Delta) - \sin p_j\Delta(1-\cos p_k\Delta), \quad \text{and}\end{aligned}$$

$$(\operatorname{Im} q_k\bar{q}_j\Delta^2)^2 = \sin^2 p_k\Delta(1-\cos p_j\Delta)^2$$
$$-2\sin p_k\Delta(1-\cos p_j\Delta)\sin p_j\Delta(1-\cos p_k\Delta)+\sin^2 p_j\Delta(1-\cos p_k\Delta)^2,$$

$$\begin{aligned}&\rho^2\Delta^4 =\\ &[(\operatorname{Im} q_2\bar{q}_3)^2 + (\operatorname{Im} q_3\bar{q}_1)^2 + (\operatorname{Im} q_1\bar{q}_2)^2]\Delta^4 = \sin^2 p_2\Delta(1-\cos p_3\Delta)^2\\ &-2\sin p_2\Delta(1-\cos p_3\Delta)\sin p_3\Delta(1-\cos p_2\Delta) + \sin^2 p_3\Delta(1-\cos p_2\Delta)^2\\ &+\sin^2 p_3\Delta(1-\cos p_1\Delta)^2 - 2\sin p_3\Delta(1-\cos p_1\Delta)\sin p_1\Delta(1-\cos p_3\Delta)\\ &+\sin^2 p_1\Delta(1-\cos p_3\Delta)^2 + \sin^2 p_1\Delta(1-\cos p_2\Delta)^2\\ &-2\sin p_1\Delta(1-\cos p_2\Delta)\sin p_2\Delta(1-\cos p_1\Delta) + \sin^2 p_2\Delta(1-\cos p_1\Delta)^2\\ &=(1-\cos p_1\Delta)^2[\sin^2 p_3\Delta + \sin^2 p_2\Delta] + (1-\cos p_2\Delta)^2[\sin^2 p_3\Delta +\\ &+\sin^2 p_1\Delta] + (1-\cos p_3\Delta)^2[\sin^2 p_2\Delta + \sin 2p_1\Delta]\\ &-2\sin p_2\Delta\sin p_3\Delta(1-\cos p_3\Delta)(1-\cos p_2\Delta)\\ &-2\sin p_3\Delta\sin p_1\Delta(1-\cos p_1\Delta)(1-\cos p_3\Delta)\\ &-2\sin p_1\Delta\sin p_2\Delta(1-\cos p_2\Delta)(1-\cos p_1\Delta).\end{aligned}$$

Since $|\sin p_\mu\Delta| \le 1$, we have

$$\begin{aligned}\rho^2\Delta^4 &\le 2\sum_{j=1}^{3}(1-\cos p_j\Delta)^2 + 2(1-\cos p_3\Delta)(1-\cos p_2\Delta)\\ &\quad + 2(1-\cos p_1\Delta)(1-\cos p_3\Delta) + 2(1-\cos p_2\Delta)(1-\cos p_1\Delta)\\ &\le \frac{1}{2}\left(\sum_{j=1}^{3}(2-2\cos p_j\Delta)\right)^2 = \frac{1}{2}(|\boldsymbol{q}|^2)^2\Delta^4.\end{aligned}$$

Thus we have $\rho \le |\boldsymbol{q}|^2/\sqrt{2}$. Since $0 \le \pm\arg B_\pm^2$,

$$0 \le \pm\arg B_\pm^2 \le \tan^{-1}\sqrt{2} \le \pi/3,\ 0 \le \pm\arg B_\pm \le \pi/6.$$

If $|\boldsymbol{p}| \le M_0$, then $|\sin p_\mu\Delta| \le |p_\mu\Delta| \le \sqrt{\pi}M_0^{-1}$, and therefore $\rho^2\Delta^4 \le \pi(|\boldsymbol{q}|^2)^2\Delta^4/(2M_0^2)$ and $\rho \le \sqrt{\pi}|\boldsymbol{q}|^2/(\sqrt{2}M_0)$. Hence $\tan(\arg B_\pm^2) \le \frac{1}{M_0}\sqrt{\pi/2}$ and thus $\arg B_\pm \approx 0$ if M is infinitely large. □

By Lemma 4.1, $B_\pm$ satisfies the conditions for the constant B in Proposition 3.1, so this proposition applies for the present case and yields

Proposition 4.2. *For the quantities $B_\pm$ introduced above we have $B_\pm \neq 0, |\arg B_\pm| \leq \pi/6$ and, for all $x \in \Gamma$,*

$$\sum_{p\in\tilde{\Gamma}} \frac{e^{ixp}}{(2-2\cos p\Delta)/\Delta^2 + B_\pm^2}\eta =$$

$$= \frac{2\pi(1+\Delta B_\pm[\sqrt{4+\Delta^2 B_\pm^2}/2 + \Delta B_\pm/2])^{-|x|/\Delta}}{B_\pm\sqrt{4+\Delta^2 B_\pm^2}}.$$

If $M, N \in {}^*\mathbb{N}$ are infinitely large numbers and $\delta = \Delta B_\pm[\sqrt{4+\Delta^2 B_\pm^2}/2 + \Delta B_\pm/2]$ is infinitesimal, then, for some $e_* \approx e$, the above sum equals

$$\frac{2\pi e_*^{-B_\pm[\sqrt{4+\Delta^2 B_\pm^2}/2+\Delta B_\pm]|x|}}{B_\pm\sqrt{4+\Delta^2 B_\pm^2}}$$

which is near $2\pi e^{-B_\pm|x|}/2B_\pm$ and less than $2\pi|2^{-B|x|}/2B_\pm|$.

Our sum is evaluated further and estimated in the next proposition.

Proposition 4.3. *Let $M, N \in {}^*\mathbb{N}$ be infinitely large numbers and $M_0 = \sqrt{M}$. If $|\boldsymbol{p}| \leq M_0$, then for $x_0 \in \Gamma$*

$$\begin{aligned}&(2\pi)^{-4}\sum_{p_0\in\tilde{\Gamma}} \frac{e^{ixp}}{(2-2\cos p_0\Delta)/\Delta^2 + B_\pm^2}\eta \\ =&(2\pi)^{-3}\frac{e^{i\boldsymbol{px}}e_*(\boldsymbol{p})^{-B_\pm[\sqrt{4+\Delta^2 B_\pm^2}/2+\Delta B_\pm/2]|x_0|}}{B_\pm\sqrt{4+\Delta^2 B_\pm^2}},\end{aligned} \tag{4.5}$$

and $e_*(\boldsymbol{p}) \approx e$. If $|\boldsymbol{p}| \geq M_0$ then

$$\begin{aligned}&\left|(2\pi)^{-4}\sum_{p_0\in\tilde{\Gamma}} \frac{e^{ixp}}{(2-2\cos p\Delta)/\Delta^2 + B_\pm^2}\eta\right| \\ \leq&(2\pi)^{-3}2^{-2M_0|x_0|/\pi}\frac{1}{4M_0/\pi}.\end{aligned} \tag{4.6}$$

Proof. If $|\boldsymbol{p}| \leq M_0$, then $\rho \leq \sqrt{\pi}|\boldsymbol{q}|^2/(\sqrt{2}M_0)$ and

$$|B_\pm^2| \leq$$
$$\frac{A^2+\sqrt{2\pi}|\boldsymbol{q}|^2/M_0}{1+\Delta} \leq A^2\frac{1+\sqrt{2\pi}/M_0}{1+\Delta} \leq \sqrt{M_0^2+\tilde{m}^2}\frac{1+\sqrt{2\pi}/M_0}{1+\Delta}.$$

This shows that $\Delta B_\pm$ and $\delta = \Delta B_\pm[\sqrt{4+\Delta^2 B_\pm^2}/2 + \Delta B_\pm/2]$ are infinitesimal. Application of Proposition 4.2 implies (4.5).

Apply Lemma 4.1 to $\arg B_\pm = \theta_\pm$. This gives $|\theta_\pm| \leq \pi/6$. For $\phi_\pm = \arg\sqrt{4+\Delta^2 B_\pm^2}$ we get $0 \leq \pm\phi_\pm < \pm\theta_\pm$ and, since

$$|B_\pm^2| \geq \frac{A^2}{1-2\tilde{m}\Delta},\ |4+\Delta^2 B_\pm^2| \geq 4+\Delta^2\frac{A^2}{1-2\tilde{m}\Delta},$$

it follows

$$|B_\pm||\sqrt{4+\Delta^2 B_\pm^2}| \geq \frac{A\sqrt{4+\Delta^2 A^2}}{1-2\tilde{m}\Delta},$$

$$\operatorname{Re} B_\pm \sqrt{4+\Delta^2 B_\pm^2} = |B_\pm||\sqrt{4+\Delta^2 B_\pm^2}| \operatorname{Re} e^{i\theta_\pm} e^{i\phi_\pm}$$

$$\geq \frac{1}{2}|B_\pm||\sqrt{4+\Delta^2 B_\pm^2}| \geq \frac{1}{2(1-2\tilde{m}\Delta)} A\sqrt{4+\Delta^2 A^2}.$$

Since (recall that $z_\pm$ is defined by Eq. (3.9) with B replaced by $B_\pm$)

$$|z_+| \geq \operatorname{Re} z_+ = \frac{2+\operatorname{Re}\Delta^2 B_\pm^2 + \operatorname{Re}\Delta B_\pm \sqrt{4+\Delta^2 B_\pm^2}}{2}$$

$$\geq 1 + (1/2)[\Delta^2 A^2 + (1/2)\Delta A\sqrt{4+\Delta^2 A^2}]/(1-2\tilde{m}\Delta),$$

we have

$$\frac{|z_+^{-|x_0|/\Delta}|}{|B_\pm\sqrt{4+\Delta^2 B_\pm^2}|} \leq \frac{\{1+(1/4)\Delta A[\sqrt{4+\Delta^2 A^2}+2\Delta A]/(1-2\tilde{m}\Delta)\}^{-|x_0|/\Delta}}{A\sqrt{4+\Delta^2 A^2}/(1-2\tilde{m}\Delta)}. \quad (4.7)$$

If $|\boldsymbol{p}| = M_0$, then $\delta = (1/4)\Delta A[\sqrt{4+\Delta^2 A^2} + 2\Delta A]/(1-2\tilde{m}\Delta)$ is infinitesimal, and the right hand side of Eq. (4.7) is:

$$\frac{\{(1+\delta)^{1/\delta}\}^{-\delta|x_0|/\Delta}}{A\sqrt{4+\Delta^2 A^2}} = \frac{e_*^{-\{(1/4)A[\sqrt{4+\Delta^2 A^2}+2\Delta A]/(1-2\tilde{m}\Delta)\}|x_0|}}{A\sqrt{4+\Delta^2 A^2}}$$

$$= \frac{(e_{**}^{1/(1-2\tilde{m}\Delta)})^{-\{(1/2)A\}|x_0|}}{2A} \leq \frac{2^{-\{(1/2)A\}|x_0|}}{2A} \leq 2^{-2M_0|x_0|/\pi}\frac{1}{4M_0/\pi},$$

where we used again the fact that

$$|\boldsymbol{q}|^2 = \sum_{\mu=1}^{3}\frac{2-2\cos p_\mu\Delta}{\Delta^2} \geq 4/\pi^2\sum_{\mu=1}^{3}|p_\mu|^2 = 4/\pi^2|\boldsymbol{p}|^2 \geq 4M_0^2/\pi^2.$$

Since the right hand side of Eq. (4.7) is a decreasing function of A, we estimate, for $|\boldsymbol{p}| \geq M_0$, as follows:

$$\frac{|z_+^{-|x_0|/\Delta}|}{|B_\pm\sqrt{4+\Delta^2 B_\pm^2}|} \leq 2^{-2M_0|x_0|/\pi}\frac{1}{4M_0/\pi}.$$

This proves (4.6). □

After these preparations, in a sequence of lemmas, the evaluation of the four dimensional lattice sum for the Dirac field is done successively. The first step is:

Lemma 4.4. *For infinitely large numbers $M, N \in {}^*\mathbb{N}$, if $x_0 \in \Gamma$ is not infinitesimal, the four dimensional lattice sum is approximated by a three dimensional one:*

$$(2\pi)^{-4} \sum_{p \in \tilde{\Gamma}^4} \frac{e^{ixp}}{(2 - 2\cos p_0 \Delta)/\Delta^2 + B_\pm^2} \eta^4$$

$$\approx (2\pi)^{-3} \sum_{\boldsymbol{p} \in \tilde{\Gamma}^3} \frac{e^{i\boldsymbol{p}\boldsymbol{x}} e^{-\sqrt{|\boldsymbol{p}|^2 + \tilde{m}^2}|x_0|}}{2\sqrt{|\boldsymbol{p}|^2 + \tilde{m}^2}} \eta^3.$$

Proof. With the abbreviation $M_0 = \sqrt{M}$ we estimate as follows:

$$\left| (2\pi)^{-4} \sum_{\boldsymbol{p} \in \tilde{\Gamma}^3, |\boldsymbol{p}| \geq M_0} \sum_{p_0 \in \tilde{\Gamma}} \frac{e^{ixp}}{(2 - 2\cos p_0 \Delta)/\Delta^2 + B_\pm^2} \eta^4 \right|$$

$$\leq (2\pi)^{-3} \sum_{\boldsymbol{p} \in \tilde{\Gamma}^3, |\boldsymbol{p}| \geq M_0} \frac{2^{-2M_0|x_0|/\pi}}{4M_0/\pi} \eta^3 \leq (2\pi)^{-3/2} M^3 \frac{2^{-2M_0|x_0|/\pi}}{4M_0/\pi} \approx 0.$$

Since

$$\left| \frac{e^{i\boldsymbol{p}\boldsymbol{x}} e_*(\boldsymbol{p})^{-B_\pm[\sqrt{4+\Delta^2 B_\pm^2}/2 + \Delta B_\pm/2]|x_0|}}{B_\pm \sqrt{4 + \Delta^2 B_\pm^2}} \right|$$

$$\leq \frac{2^{-A|x_0|}}{2A} \leq \frac{2^{-2|\boldsymbol{p}||x_0|/\pi}}{4|\boldsymbol{p}|/\pi}$$

for $|\boldsymbol{p}| \leq M_0$, for any standard $\epsilon > 0$, there exists a finite $M_1 > 0$ such that

$$\left| (2\pi)^{-3} \sum_{\boldsymbol{p} \in \tilde{\Gamma}^3, M_1 \leq |\boldsymbol{p}| \leq M_0} \frac{e^{i\boldsymbol{p}\boldsymbol{x}} e_*(\boldsymbol{p})^{-B_\pm[\sqrt{4+\Delta^2 B_\pm^2}/2 + \Delta B_\pm/2]|x_0|}}{B_\pm \sqrt{4 + \Delta^2 B_\pm^2}} \eta^3 \right| < \epsilon$$

if $|x_0|$ is not infinitesimal. This shows that

$$\forall\, \epsilon > 0\ \exists\, M_1 \left| (2\pi)^{-4} \sum_{\boldsymbol{p} \in \tilde{\Gamma}^3, M_1 \leq |\boldsymbol{p}|} \sum_{p_0 \in \tilde{\Gamma}} \frac{e^{ixp}}{(2 - 2\cos p_0 \Delta)/\Delta^2 + B_\pm^2} \eta^4 \right| < \epsilon.$$

We also know

$$\forall\, \epsilon > 0\ \exists\, M_1 \left| (2\pi)^{-3} \sum_{\boldsymbol{p} \in \tilde{\Gamma}^3, |\boldsymbol{p}| \geq M_1} \frac{e^{-\sqrt{|\boldsymbol{p}|^2 + \tilde{m}^2}|x_0|}}{\sqrt{|\boldsymbol{p}|^2 + \tilde{m}^2}} \eta^3 \right| < \epsilon.$$

If $|\boldsymbol{p}| < M_1$ for a finite $M_1 > 0$, then it follows that

$$\rho \approx 0,\ B_\pm \approx A \approx \sqrt{\boldsymbol{p}^2 + \tilde{m}^2}$$

and

$$\frac{e_*(\boldsymbol{p})^{-B_\pm[\sqrt{4+\Delta^2 B_\pm^2}/2 + \Delta B_\pm/2]|x_0|}}{B_\pm \sqrt{4 + \Delta^2 B_\pm^2}} \approx \frac{e^{-\sqrt{|\boldsymbol{p}|^2 + \tilde{m}^2}|x_0|}}{2\sqrt{|\boldsymbol{p}|^2 + \tilde{m}^2}}.$$

Hence, for non-infinitesimal $|x_0|$, we conclude

$$(2\pi)^{-3} \sum_{\boldsymbol{p}\in\tilde{\Gamma}^3, |\boldsymbol{p}|\le M_1} \frac{e^{i\boldsymbol{p}\boldsymbol{x}} e_*(\boldsymbol{p})^{-B_\pm[\sqrt{4+\Delta^2 B_\pm^2}/2+\Delta B_\pm/2]|x_0|}}{B_\pm\sqrt{4+\Delta^2 B_\pm^2}} \eta^3$$

$$\approx (2\pi)^{-3} \sum_{\boldsymbol{p}\in\tilde{\Gamma}^3, |\boldsymbol{p}|\le M_1} \frac{e^{i\boldsymbol{p}\boldsymbol{x}} e^{-\sqrt{|\boldsymbol{p}|^2+\tilde{m}^2}|x_0|}}{2\sqrt{|\boldsymbol{p}|^2+\tilde{m}^2}} \eta^3$$

and the proof is complete. □

Lemma 4.5. *Under the same condition as Lemma 4.4 the four dimensional lattice sum*

$$(2\pi)^{-4} \sum_{p\in\tilde{\Gamma}^4} \frac{e^{ixp} q_k}{(2-2\cos p_0\Delta)/\Delta^2 + B_\pm^2} \eta^4$$

is infinitesimally close to the spatial three dimensional one

$$\approx (2\pi)^{-3} \sum_{\boldsymbol{p}\in\tilde{\Gamma}^3} \frac{e^{i\boldsymbol{p}\boldsymbol{x}} p_k e^{-\sqrt{|\boldsymbol{p}|^2+\tilde{m}^2}|x_0|}}{2\sqrt{|\boldsymbol{p}|^2+\tilde{m}^2}} \eta^3$$

for $k = 1, 2, 3$.

Proof. We can proceed in the same way as in the proof of Lemma 4.4. □

Lemma 4.6. *Under the same condition as Lemma 4.4, the four dimensional lattice sum for the Dirac operator is reduced to a three dimensional one in the following way:*

$$(2\pi)^{-4} \sum_{p\in\tilde{\Gamma}^4} \frac{e^{ixp}}{\kappa^2+4\rho^2} \begin{pmatrix} \tilde{m} & -\boldsymbol{\sigma}\cdot\boldsymbol{q} \\ \boldsymbol{\sigma}\cdot\overline{\boldsymbol{q}} & \tilde{m} \end{pmatrix} \begin{pmatrix} K^* & 0 \\ 0 & K \end{pmatrix} \eta^4$$

$$\approx (2\pi)^{-3} \sum_{\boldsymbol{p}\in\tilde{\Gamma}^3} \frac{e^{i\boldsymbol{p}\boldsymbol{x}} e^{-\sqrt{|\boldsymbol{p}|^2+\tilde{m}^2}|x_0|}}{2\sqrt{|\boldsymbol{p}|^2+\tilde{m}^2}} \begin{pmatrix} \tilde{m} & -\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}} & \tilde{m} \end{pmatrix} \eta^3.$$

Proof. Note that if $\rho \neq 0$

$$\sum_{p_0\in\tilde{\Gamma}} \frac{e^{ix_0p_0}}{\kappa^2+4\rho^2} K\eta = \sum_{p_0\in\tilde{\Gamma}} \frac{e^{ix_0p_0}}{\kappa^2+4\rho^2} (\kappa\sigma_0 + 2iE)\eta$$

$$= \sum_{p_0\in\tilde{\Gamma}} e^{ix_0p_0} \frac{1}{2} \left(\frac{1}{\kappa-2i\rho} + \frac{1}{\kappa+2i\rho} \right) \sigma_0\eta$$

$$+ \sum_{p_0\in\tilde{\Gamma}} e^{ix_0p_0} \frac{1}{2\rho} \left(\frac{1}{\kappa-2i\rho} - \frac{1}{\kappa+2i\rho} \right) E\eta,$$

and

$$\sum_{p_0\in\tilde{\Gamma}} \frac{e^{ix_0p_0}}{\kappa^2+4\rho^2} K\eta = \sum_{p_0\in\tilde{\Gamma}} \frac{e^{ix_0p_0}}{\kappa} \sigma_0\eta$$

if $\rho = 0$. Since

$$\sum_{p_0 \in \tilde{\Gamma}} e^{ix_0 p_0} \frac{1}{\kappa \pm 2i\rho} \eta = \frac{1}{1+\Delta} \sum_{p_0 \in \tilde{\Gamma}} \frac{e^{ix_0 p_0}}{(2 - 2\cos p_0 \Delta)/\Delta^2 + B_\pm^2} \eta$$

$$= \frac{2\pi}{1+\Delta} \frac{(1 + \Delta B_\pm [\sqrt{4 + \Delta^2 B_\pm^2}/2 + \Delta B_\pm/2])^{|x_0|/\Delta}}{B_\pm \sqrt{4 + \Delta^2 B_\pm^2}}$$

and E/ρ is finite, this lemma follows from Lemmas 4.4 and 4.5. $\square$

Lemma 4.7. *Under the same condition as Lemma 4.4 the following approximation holds:*

$$(2\pi)^{-4} \sum_{p \in \tilde{\Gamma}^4} \frac{i\bar{q}_0 e^{ixp}}{(2 - 2\cos p_0 \Delta)/\Delta^2 + B_\pm^2} \eta^4$$

$$\approx -\frac{x^0}{|x^0|} (2\pi)^{-3} \sum_{\boldsymbol{p} \in \tilde{\Gamma}^3} \frac{e^{i\boldsymbol{p}\boldsymbol{x}} e^{-\sqrt{|\boldsymbol{p}|^2 + \tilde{m}^2}|x^0|}}{2} \eta^3.$$

Proof.

$$\sum_{p_0 \in \tilde{\Gamma}} \frac{i\bar{q}_0 e^{ix_0 p_0}}{(2 - 2\cos p\Delta)/\Delta^2 + B_\pm^2} \eta = \sum_{p_0 \in \tilde{\Gamma}} \frac{\nabla^+ e^{ix_0 p_0}}{(2 - 2\cos p\Delta)/\Delta^2 + B_\pm^2} \eta$$

$$= \frac{2\pi \nabla^+ z_-^{x_0/\Delta}}{z_+ - z_-} = \frac{2\pi z_-^{x_0/\Delta}(z_- - 1)/\Delta}{z_+ - z_-}$$

$$= \frac{2\pi z_-^{x_0/\Delta}}{z_+ - z_-} \frac{\Delta^2 B_\pm^2 - B_\pm \sqrt{4 + \Delta^2 B_\pm^2}}{2}$$

if $x_0 \geq 0$, and if $x_0 < 0$, then

$$= \frac{2\pi z_+^{x_0/\Delta}}{z_+ - z_-} \frac{\Delta^2 B^2 + B\sqrt{4 + \Delta^2 B^2}}{2}.$$

Now we can prove this lemma in the same way as Lemma 4.4, since

$$\frac{\Delta^2 B_\pm^2 \pm \sqrt{4 + \Delta^2 B_\pm^2}}{2\sqrt{4 + \Delta^2 B^2}} \approx \pm\frac{1}{2}$$

for $|\boldsymbol{p}| \leq M_0$. $\square$

Lemma 4.8. *Under the same condition as Lemma 4.4 the following approximation*

$$(2\pi)^{-4} \sum_{p \in \tilde{\Gamma}^4} \frac{e^{ixp}}{\kappa^2 + 4\rho^2} \begin{pmatrix} -iq_0 & 0 \\ 0 & i\bar{q}_0 \end{pmatrix} \begin{pmatrix} K^* & 0 \\ 0 & K \end{pmatrix} \eta^4$$

$$\approx \frac{x_0}{|x_0|} \gamma_0^E (2\pi)^{-3} \sum_{\boldsymbol{p} \in \tilde{\Gamma}^3} \frac{e^{i\boldsymbol{p}\boldsymbol{x}} e^{-\sqrt{|\boldsymbol{p}|^2 + \tilde{m}^2}|x_0|}}{2} \eta^3$$

of the four dimensional by a three dimensional lattice sum holds.

Proof. We can prove this lemma in the same way as Lemma 4.5. □

The combination of lemmas 4.6 and 4.8 gives

Lemma 4.9. *Under the same condition as Lemma 4.4, the four dimensional lattice sum for the Dirac operator is reduced to a three dimensional one:*

$$(2\pi)^{-4} \sum_{p\in\tilde{\Gamma}^4} \frac{e^{ixp}}{\kappa^2+4\rho^2} \begin{pmatrix} -iq_0+\tilde{m} & -\boldsymbol{\sigma}\cdot\boldsymbol{q} \\ \boldsymbol{\sigma}\cdot\overline{\boldsymbol{q}} & i\bar{q}_0+\tilde{m} \end{pmatrix} \begin{pmatrix} K^* & 0 \\ 0 & K \end{pmatrix} \eta^4$$

$$\approx (2\pi)^{-3} \sum_{\boldsymbol{p}\in\tilde{\Gamma}^3} \frac{e^{i\boldsymbol{p}\boldsymbol{x}} e^{-\sqrt{|\boldsymbol{p}|^2+\tilde{m}^2}|x_0|}}{2\sqrt{|\boldsymbol{p}|^2+\tilde{m}^2}} \begin{pmatrix} \tilde{m} & -\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}} & \tilde{m} \end{pmatrix} \eta^3$$

$$+\frac{x_0}{|x_0|}\gamma_0^E (2\pi)^{-3} \sum_{\boldsymbol{p}\in\tilde{\Gamma}^3} \frac{e^{i\boldsymbol{p}\boldsymbol{x}} e^{-\sqrt{|\boldsymbol{p}|^2+\tilde{m}^2}|x_0|}}{2} \eta^3.$$

Lemma 4.10. *Assume that $M, N \in {}^*\mathbb{N}$ are infinitely large numbers. If $\boldsymbol{x}$ is finite and $|x_0|$ not infinitesimal, then the 'spatial' lattice sum is approximated by the expected integral:*

$$(2\pi)^{-3} \sum_{\boldsymbol{p}\in\tilde{\Gamma}^3} \frac{e^{i\boldsymbol{p}\boldsymbol{x}} e^{-\sqrt{|\boldsymbol{p}|^2+\tilde{m}^2}|x_0|}}{2\sqrt{|\boldsymbol{p}|^2+\tilde{m}^2}} \begin{pmatrix} \tilde{m} & -\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}} & \tilde{m} \end{pmatrix} \eta^3$$

$$+\frac{x_0}{|x_0|}\gamma_0^E (2\pi)^{-3} \sum_{\boldsymbol{p}\in\tilde{\Gamma}^3} \frac{e^{i\boldsymbol{p}\boldsymbol{x}} e^{-\sqrt{|\boldsymbol{p}|^2+\tilde{m}^2}|x_0|}}{2} \eta^3.$$

$$\approx (2\pi)^{-3} \int_{{}^*\mathbb{R}^3} \frac{e^{i\boldsymbol{p}\boldsymbol{x}} e^{-\sqrt{|\boldsymbol{p}|^2+\tilde{m}^2}|x_0|}}{2\sqrt{|\boldsymbol{p}|^2+\tilde{m}^2}} \begin{pmatrix} \tilde{m} & -\boldsymbol{\sigma}\cdot\boldsymbol{p} \\ \boldsymbol{\sigma}\cdot\overline{\boldsymbol{p}} & \tilde{m} \end{pmatrix} \eta^3 d\boldsymbol{p}$$

$$+\frac{x_0}{|x_0|}\gamma_0^E (2\pi)^{-3} \int_{{}^*\mathbb{R}^3} \frac{e^{i\boldsymbol{p}\boldsymbol{x}} e^{-\sqrt{|\boldsymbol{p}|^2+\tilde{m}^2}|x_0|}}{2} \eta^3 d\boldsymbol{p}.$$

Proof. The proof strategy of Lemma 3.5 applies. □

Using the formulae ($A > 0$)

$$(2\pi)^{-1} \int_{\mathbb{R}} \frac{e^{ipx}}{p_0^2+A^2} dp_0 = \frac{e^{i\boldsymbol{p}\boldsymbol{x}} e^{-A|x_0|}}{2A},$$

$$(2\pi)^{-1} \int_{\mathbb{R}} \frac{-ip_0 e^{ipx}}{p_0^2+A^2} dp_0 = \frac{x_0}{|x_0|} \frac{e^{i\boldsymbol{p}\boldsymbol{x}} e^{-A|x_0|}}{2},$$

we get the main result of this section.

Theorem 4.11. *For infinitely large numbers $M, N \in {}^*\mathbb{N}$, for all finite $x \in \Gamma^4$ for which $|x_0|$ is not infinitesimal, the lattice Fourier transform of (4.3) is given by*

$$(4\pi)^{-4} \sum_{p \in \tilde{\Gamma}^4} e^{ipx} \begin{pmatrix} i\bar{q}_0 + \tilde{m} & \boldsymbol{\sigma} \cdot \boldsymbol{q} \\ -\boldsymbol{\sigma} \cdot \overline{\boldsymbol{q}} & -iq_0 + \tilde{m} \end{pmatrix}^{-1} \eta^4 =$$

$$= (2\pi)^{-4} \sum_{p \in \tilde{\Gamma}^4} \frac{e^{ipx}}{\kappa^2 + 4\rho^2} \left(\begin{array}{cc} -iq_0 + \tilde{m} & -\boldsymbol{\sigma} \cdot \boldsymbol{q} \\ \boldsymbol{\sigma} \cdot \overline{\boldsymbol{q}} & i\bar{q}_0 + \tilde{m} \end{array} \right) \left(\begin{array}{cc} K^* & 0 \\ 0 & K \end{array} \right) \eta^4$$

$$\approx (2\pi)^{-4} \int_{{}^*\mathbb{R}^4} \frac{e^{ipx}}{p^2 + m^2} \left(\begin{array}{cc} -ip_0 + \tilde{m} & -\boldsymbol{\sigma} \cdot \boldsymbol{p} \\ \boldsymbol{\sigma} \cdot \boldsymbol{p} & ip_0 + \tilde{m} \end{array} \right) dp. \tag{4.8}$$

5. Convergence of the lattice approximation for the interacting theory – in the sense of ultrahyperfunctions

For a motivation of relativistic quantum field theory in terms of tempered ultra-hyperfunctions as the appropriate framework for a relativistic quantum field theory with a fundamental length and for a brief introduction to the mathematics of such a theory we have to refer to [2]. Here we just mention the basic definitions and results about tempered ultra-hyperfunctions as we need them.

For a subset A of $\mathbb{R}^n$, we denote by $T(A) = \mathbb{R}^n + iA \subset \mathbb{C}^n$ the tubular set with base A. For a convex compact set K of $\mathbb{R}^n$, $\mathcal{T}_b(T(K))$ is, by definition, the space of all continuous functions f on $T(K)$ which are holomorphic in the interior of $T(K)$ and satisfy

$$\|f\|^{T(K),j} = \sup\{|z^p f(z)|; z \in T(K), |p| \leq j\} < \infty, \; j = 0, 1, \ldots$$

where $p = (p_1, \ldots, p_n)$ and $z^p = z_1^{p_1} \cdots z_n^{p_n}$. $\mathcal{T}_b(T(K))$ is a Fréchet space with the semi-norms $\|f\|^{T(K),j}$. If $K_1 \subset K_2$ are two compact convex sets, we have the canonical injections:

$$\mathcal{T}_b(T(K_2)) \to \mathcal{T}_b(T(K_1)).$$

Let O be a convex open set in $\mathbb{R}^n$. We define

$$\mathcal{T}(T(O)) = \lim_{\leftarrow} \mathcal{T}_b(T(K_1)),$$

where K_1 runs through the convex compact sets contained in O and the projective limit is taken following the restriction mappings.

Definition 5.1. *A tempered ultra-hyperfunction is by definition a continuous linear functional on $\mathcal{T}(T(\mathbb{R}^n))$.*

Characterizations of tempered ultra-hyperfunctions are known since many years ([11, 16, 17]). The most convenient one for our purposes is based on a result in [2] which we prepare briefly.

Let $\mathcal{A}_0(W)$ be the space of all functions F, holomorphic in an open set $W \subset \mathbb{C}^n$, with the property that for any positive numbers ϵ, K, there exist a multi-index p and a constant $C \geq 0$ such that

$$|F(z)| \leq C(1+|z^p|) \quad \text{for all } \ z \in \mathbb{C}^n \backslash (\mathbb{C}^n \backslash W)_\epsilon, \ |\mathrm{Im}\, z_j| \leq K$$

where $(\mathbb{C}^n \backslash W)_\epsilon$ is the open $\epsilon-$neighbourhood of $(\mathbb{C}^n \backslash W)$. Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be a vector with components $\sigma_j \in \{\pm 1\}$. For such a vector σ and a number $k > 0$ introduce the open set

$$\mathbb{C}^n_{\sigma,k} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n; \sigma_j \mathrm{Im}\, z_j > k, \text{ for } j = 1, \ldots, n\}$$

and the space $\mathcal{A}_0(\mathbb{C}^n_{\sigma,k})$ introduced above. Next we consider collections $\{F_\sigma\}$ of elements $F_\sigma \in \mathcal{A}_0(\mathbb{C}^n_{\sigma,k})$. Furthermore, for $\epsilon > 0$, $k > 0$, and $\sigma_j \in \{\pm 1\}$, define the path

$$\Gamma_{\sigma_j} \equiv \Gamma_{\sigma_j}(\epsilon, k) \stackrel{\text{def}}{=} \{z \in \mathbb{C}; z = x + i\sigma_j(k+\epsilon), x \in \mathbb{R}\}.$$

and then the product path $\Gamma_\sigma = \prod_{j=1}^n \Gamma_{\sigma_j}$.

Then from the definition of the spaces $\mathcal{T}(T(\mathbb{R}^n))$ and $\mathcal{A}_0(\mathbb{C}^n_{\sigma,k})$ it is clear that for any collection $\{F_\sigma\}$ of $F_\sigma \in \mathcal{A}_0(\mathbb{C}^n_{\sigma,k})$ the assignment

$$\mathcal{T}(T(\mathbb{R}^n)) \ni f \to \langle \{F_\sigma\}, f\rangle = \sum_\sigma \sigma_1 \cdots \sigma_n \int_{\Gamma_\sigma} F_\sigma(z) f(z) dz \in \mathbb{C} \quad (5.1)$$

is well defined and for fixed collection $\{F_\sigma\}$ is linear and continuous in $f \in \mathcal{T}(T(\mathbb{R}^n))$. Thus for given collection $\{F_\sigma\}$, $F_\sigma \in \mathcal{A}_0(\mathbb{C}^n_{\sigma,k})$,

$$\mathcal{T}(T(K)) \ni f \to \langle \{F_\sigma\}, f\rangle$$

is a tempered ultra-hyperfunction. Conversely, it is shown in [11, 2] that for any element M of $\mathcal{T}(T(\mathbb{R}^n))'$, there exist constant $k > 0$ and a collection $\{F_\sigma\}$ of functions F_σ in $\mathcal{A}_0(\mathbb{C}^n_{\sigma,k})$ such that

$$M(f) = \langle \{F_\sigma\}, f\rangle \quad (5.2)$$

for all $f \in \mathcal{T}(T(\mathbb{R}^n))$ (see also [11, 16, 17]). This proves

Theorem 5.2 (characterization tempered ultra-hyperfunctions). *A linear functional M on $\mathcal{T}(T(\mathbb{R}^n))$ is a tempered ultra-hyperfunction if, and only if, it is of the form (5.1), (5.2) for some $k > 0$ and some collection $\{F_\sigma\}$ of functions F_σ in $\mathcal{A}_0(\mathbb{C}^n_{\sigma,k})$.*

Remark 5.3. In quantum field theory with a fundamental length, often functionals appear which are defined for $g \in \mathcal{T}(T(\mathbb{R}^{2\cdot 4}))$ by

$$\begin{aligned}\langle F, g\rangle &= \int F(x_1^0 - x_2^0 - i(k+\epsilon), \boldsymbol{x}_1 - \boldsymbol{x}_2) g(x_1^0 - i(k+\epsilon), \boldsymbol{x}_1, x_2^0, \boldsymbol{x}_2) dx_1^0 \cdots dx_2^3 \\ &= \int F(x^0 - i(k+\epsilon), \boldsymbol{x}) f(x^0 - i(k+\epsilon), \boldsymbol{x}) dx^0 \cdots dx^3 \qquad (5.3)\end{aligned}$$

for an analytic function $F \in \mathcal{A}_0(W)$ defined in the region

$$W = \{z = (z^0, \ldots, z^3) \in \mathbb{C}^4; -\mathrm{Im}\, z^0 > |\mathrm{Im}\, \boldsymbol{z}| + k\}$$

and

$$f(z) = \int g(z + x_2, x_2) dx_2^0 \cdots dx_2^3 \in \mathcal{T}(T(\mathbb{R}^4)).$$

It is clear that the integral (5.3) defines a tempered ultra-hyperfunction. But the integral representation (5.3) looks quite different from the integral representation (5.1), (5.2) which characterizes tempered ultra-hyperfunctions M according to Theorem 5.2. Here we explain that the integral representation (5.1), (5.2) can be expressed by the integral (5.3) in certain situations (e.g., the support of the Fourier transformation $\tilde{M}$ of M is contained in the forward light-cone $\bar{V}_+ = \{x \in \mathbb{R}^4; x^0 \geq |\boldsymbol{x}|\}$). For simplicity, we assume $n = 2$. Consider the situation that $F_{(1,1)} = F_{(-1,1)} = 0$. Then

$$\langle F, f\rangle = \sum_{\sigma} \sigma_1\sigma_2 \int_{\Gamma_\sigma} F_\sigma(z) f(z) dz = \sum_{\sigma=(1,-1),(-1,-1)} \sigma_1\sigma_2 \int_{\Gamma_\sigma} F_\sigma(z) f(z) dz$$

$$= -\int\int F_{(1,-1)}(x_1+i(k+\epsilon), x_2-i(k+\epsilon)) f(x_1+i(k+\epsilon), x_2-i(k+\epsilon)) dx_1 dx_2$$

$$+ \int\int F_{(-1,-1)}(x_1-i(k+\epsilon), x_2-i(k+\epsilon)) f(x_1-i(k+\epsilon), x_2-i(k+\epsilon)) dx_1 dx_2.$$

Now we further assume that $F_{(\pm1,-1)}$ is analytically continued from

$$\mathbb{C}^2_{(\pm1,-1),k} = \{(z_1, z_2) \in \mathbb{C}^2; \pm\mathrm{Im}\, z_1 > k,\ -\mathrm{Im}\, z_2 > k\}$$

to the set

$$\{(z_1, z_2) \in \mathbb{C}^2; \pm\mathrm{Im}\, z_1 - \mathrm{Im}\, z_2 > 2k,\ -\mathrm{Im}\, z_2 > k\}.$$

Then by deforming the path of integration, we get

$$\int\int F_{(1,-1)}(x_1+i(k+\epsilon), x_2-i(k+\epsilon)) f(x_1+i(k+\epsilon), x_2-i(k+\epsilon)) dx_1 dx_2$$

$$= \int\int F_{(1,-1)}(x_1, x_2 - i(2k+\epsilon)) f(x_1, x_2 - i(2k+\epsilon)) dx_1 dx_2$$

and

$$\int\int F_{(-1,-1)}(x_1-i(k+\epsilon), x_2-i(k+\epsilon)) f(x_1-i(k+\epsilon), x_2-i(k+\epsilon)) dx_1 dx_2$$

$$= \int\int F_{(-1,1)}(x_1, x_2 - i(2k+\epsilon)) f(x_1, x_2 - i(2k+\epsilon)) dx_1 dx_2.$$

Put $G(z_1, z_2) = -F_{(1,-1)}(z_1, z_2) - F_{(-1,1)}(z_1, z_2)$. Then $G(z_1, z_2)$ is analytic in

$$\{(z_1, z_2) \in \mathbb{C}^2; (\mathrm{Im}\, z_1 - \mathrm{Im}, z_2 > 2k) \wedge (-\mathrm{Im}\, z_1 - \mathrm{Im}\, z_2 > 2k)\}$$
$$= \{(z_1, z_2) \in \mathbb{C}^2; -\mathrm{Im}\, z_2 > |\mathrm{Im}\, z_1| + 2k\},$$

and we have

$$\langle F, f\rangle = \int\int G(x_1, x_2 + i(2k+\epsilon)) f(x_1, x_2 + i(2k+\epsilon)) dx_1 dx_2.$$

As we had seen at the end of Section 2, the n-point Wightman function of the field $\psi(x)$ is

$$\mathcal{W}^r_\alpha(x_1, \dots, x_n) = (\det A)^{-1/2}\mathcal{W}^r_{0,\alpha}(x_1, \dots, x_n), \tag{5.4}$$

where A is the matrix determined by (2.9), i.e., $(a_{j,k}),\ j,k = 1, \cdots, n$ with

$$a_{j,j} = 1,\ a_{j,k} = a_{k,j} = 2h_{r_j}h_{r_k}l^2 D^{(-)}_m(x_j - x_k), \text{if } j < k.$$

In [2] the functional characterization of a relativistic quantum field theory with a fundamental length has been given in terms of six conditions (R0) $\cdots$ (R5). Now we are going to show that the system (5.4) satisfies condition (R0) which states that this systems consists of symmetric tempered ultra-hyperfunctions. The first part of this condition (R0) says that the assignment

$$\mathcal{T}(T(\mathbb{R}^{4n}) \ni f \to \mathcal{W}^{r_1,\dots,r_n}_{\alpha_1,\dots,\alpha_n}(f) \in \mathbb{C}$$

is a continuous linear functional on $\mathcal{T}(T(\mathbb{R}^{4n}))$, for $n = 1, 2, 3, \dots$.

In order to investigate this continuity property, we apply the general expansion formula for determinants we get

$$\det A = \sum \operatorname{sgn}(j, k, \dots, l) a_{1,j}a_{2,k}\cdots a_{n,l}$$

$$= a_{1,1}a_{2,2}\cdots a_{n,n} + \sum_{(j,k,\dots,l)\neq(1,2,\dots,n)} \operatorname{sgn}(j, k, \dots, l) a_{1,j}a_{2,k}\cdots a_{n,l}.$$

Because of the special values of the entries $a_{j,k}$ according to (2.9) we see

$$\det A = 1 + P_n(a_{j,k}) \tag{5.5}$$

where $P_n(a_{j,k})$ is the sum of homogeneous polynomials of degrees $m = 2, \cdots, n$ in the entries $a_{j,k},\ 1 \le j < k \le n$ with integer coefficients. The integral representation for $D^{(-)}_m$ as given at the end of Section 3 easily implies, for every $\epsilon > 0$, the global estimate

$$|D^{(-)}_m(x^0 - i\epsilon, \boldsymbol{x})| \le (2\pi\epsilon)^{-2} \quad \text{for all } x \in \mathbb{R}^4. \tag{5.6}$$

It follows that $|P(a_{j,k})| < 1$ if we choose all $y^0_k - y^0_j,\ j < k$, sufficiently large and put $z_j = (x^0_j + iy^0_j, \boldsymbol{x}_j)$. Hence for these z_j, $(\det A(z))^{-1/2} = (1 + P(a_{j,k}(z_j, z_k)))^{-1/2}$ is a bounded analytic function of the x_j in a tubular domain and therefore, according to Theorem 5.2,

$$\mathcal{W}^r_\alpha(z_1, \dots, z_n) = (\det A(z))^{-1/2}\mathcal{W}^r_{0,\alpha}(z_1, \dots, z_n)$$

determines a tempered ultra-hyperfunction by the formula

$$\mathcal{W}^r_\alpha(f) = \int_{\prod_{j=1}^n \Gamma_j} (\det A(z))^{-1/2}\mathcal{W}^r_{0,\alpha}(z_1, \dots, z_n) f(z) dz, \tag{5.7}$$

where $\Gamma_j = \mathbb{R}^4 + i(y^0_j, 0, 0, 0)$, for all $f \in \mathcal{T}(T(\mathbb{R}^{4n}))$, i.e., the first part of Condition (R0) is satisfied.

We conclude that the sequence of Wightman functions $\mathcal{W}^r_\alpha$ satisfies $\mathcal{W}^r_\alpha \in \mathcal{T}(T(\mathbb{R}^{4n}))'$ for $n = 1, 2, 3, \ldots$. The second part of Condition (R0), i.e.,

$$\mathcal{W}^{\bar{r}_n,\ldots,\bar{r}_1}_{\bar{\alpha}_n,\ldots,\bar{\alpha}_1}(f^*) = \overline{\mathcal{W}^{r_1,\ldots,r_n}_{\alpha_1,\ldots,\alpha_n}(f)},\ f^*(z_1,\ldots,z_n) = \overline{f(\overline{z}_n,\ldots,\overline{z}_1)},$$

where $\psi^{r*}_\alpha = \psi^{\bar{r}}_{\bar{\alpha}}$, follows easily from the fact that

$$\overline{D^{(-)}_m(z_j - z_k)} = D^{(-)}_m(\overline{z}_k - \overline{z}_j).$$

To conclude this section, we have a closer look at the two-point function

$$\mathcal{W}^{1,2}_{\alpha_1,\alpha_2}(z_1, z_2) = [1 - 4l^4 D_m(z_1 - z_2)^2]^{-1/2}\mathcal{W}^{1,2}_{0,\alpha_1,\alpha_2}(z_1, z_2).$$

Estimate (5.6) shows that $|4l^4 D_m(x^0_1 - x^0_2 - i\epsilon, \boldsymbol{x}_1 - \boldsymbol{x}_2)^2| < 1$ if $\epsilon > \ell = l/(\sqrt{2}\pi)$, and $[1 - 4l^4 D_m(x^0_1 - x^0_2 - i\epsilon, \boldsymbol{x}_1 - \boldsymbol{x}_2)^2]^{-1/2}$ is analytic with respect to x_1 and x_2. Then the functional defined by (5.7) for $n = 2$ and $y_2 - y_1 = \epsilon > 0$ can distinguish the two events only if their distance is greater than ϵ (see [2]). Since $\epsilon > \ell$ is arbitrary, ℓ is the fundamental length of our theory.

6. Conclusion

The results of this article provide a solution of the linearized version of Heisenberg's fundamental equation, on the level of all the n-point functionals of the solution fields. This has been achieved by employing path integral methods for quantization. In order to have all the path integrals well defined and to evaluate them rigorously, a lattice approximation was used and the continuum limit of this approximation was controlled by using non-standard analysis. This continuum limit exists in the framework of tempered ultrahyperfunctions but not in the framework of tempered distributions. In this way in particular the convergence of the lattice approximations for a free scalar field, a free Dirac field and for the interacting fields of this model has been established.

In the second part we are going to show that the sequence of all n-point functionals which we have constructed satisfy all the defining conditions of a relativistic quantum field theory with a fundamental length, in the sense of [2]. We do so by first extending the theory of [2] to include scalar as well as spinor fields and then verifying the defining condition. In addition we offer an alternative way to calculate all the n-point functionals of the theory by use of Wick power series which converge in the sense of tempered ultrahyperfunctions. And its is shown that the solution fields (ϕ, ψ) of (1.4) - (1.5) can be express of a point-wise product

$$\psi(x) = \psi_0(x) : e^{il^2\phi(x)^2} : \tag{6.1}$$

where ψ_0 is the free Dirac field.

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